Bounding High Dimensional Comparative Statics

Jordan J. Norris*

June, 2025

Abstract

Comparative statistics in high dimensional models are empirically demanding and analytically complicated. Complete identification of the correspondingly many models parameters is required, which is often infeasible due to data availability. I derive novel, sharp bounds on high dimensional comparative statics that depend only on low dimensional sufficient statistics, the knowledge of which is often more feasible, and have a simpler functional form relative to the exact relationship. I demonstrate application in canonical models across economics, and address existing methodological limitations in the research on peer effects, the gains from trade, and price-cost passthrough.

Keywords: comparative statics, diagonal dominance, networks, partial identification, sufficient statistics
JEL Classification: C3, D85

*Norris: New York University Abu Dhabi, UAE (email: jjnorris@nyu.edu). I would like to thank Treb Allen, Ivan Canay, Jonathan Dingel, Ben Golub, Sanjeev Goyal, David Nagy, Kevin O'Rourke, Michael Reiter, Alireza Tahbaz-Salehi and seminar attendees at Aarhus University, the Bank of England, Bocconi University, Dartmouth, Drexel, Georgetown, the Inter-American Development Bank, Northwestern, NYU Abu Dhabi, NYU Stern, Osaka University, the University of Bristol, the University of Cambridge, the University of Copenhagen, the University of Queensland, the University of Virginia, UPF/CREI, UNSW, UT Austin, Warwick University and William & Mary College, and attendees at the Annual Conference on Network Science and Economics. I would especially like to thank Charlie Johnson and Ilya Spitovsky for extensive discussions on these topics.

1 Introduction

A core purpose of modeling in economics is comparative statics: the effect of an exogenous shock on an endogenous state (such as student performance, GDP, or prices). Realistic economic models tend to be *high dimensional*, having many endogenous states that interact in highly heterogeneous ways (a social network of students). A key empirical challenge is that the comparative static of even a single state requires complete knowledge of all model parameters (the entire network), as all states in general respond to changes in one another (peer effects). This is often infeasible due to data limitations (complete network data typically doesn't exist, Bramoullé et al., 2020). Moreover, even if one did have sufficient data, the functional form of the comparative static is extremely complex due to the interaction of all the states, obscuring the economics and mechanisms, which is often of interest (highlighted, for example, by Baqaee and Farhi, 2024; Bilbiie, 2018; Thisse, 2010).

In this paper, I develop a new tool to help solve these challenges. I derive *sharp* upper and lower bounds on (the linearized) comparative statics. The bounds do not require complete knowledge of all model parameters, instead depend only on a set of *low dimensional sufficient statistics* (e.g. a student's number of friends), the knowledge of which is often more feasible. The functional form of this dependence is also much simpler relative to the exact comparative static, permitting new theoretical insights.[1] The trade-off of is that the comparative static is only partially identified — a bound — with the width of the bound depending on the value of the sufficient statistics, therefore its practical usefulness depends on the application. Nonetheless, because I prove the bounds are sharp, if one were to only know the values of the sufficient statistics, the bounds are the most one can say about the maximum and minimum values the comparative static can take.

The comparative static bounds are valid in many high dimensional, canonical models across economics (see table 1). The only substantive assumption imposed is that the Jacobian (with respect to the endogenous states) is diagonally dominant, a condition often invoked for sufficiency of equilibrium uniqueness and hence its preva-

[1] Although not focused on in this paper, another advantage of the bounds is that they are less computationally costly to compute, as no matrix inversion of the Jacobian is required. This computational limitation is becoming increasingly important as increasingly granular data are utilized, such as in the calculation of Leontief inverses on massive firm-firm transaction datasets (Carvalho et al., 2021; Fujiy et al., 2024).

lence. Diagonal dominance can be understood as the feedback in the system being greater within a state than between states (Arrow and Hahn, 1971 pg 233: a product is more sensitive to changes in its own price than the prices of all other products combined). Notably, the number of states is not restricted, and neither is their heterogeneity beyond this assumption, thus permitting its application in high dimensional models. I derive variations that exploit instead the spectral radius of a suitably transformed Jacobian (coinciding with the adjacency matrix in social network models, of which there is already much interest, Bramoullé et al. 2014; Golub 2025), or the sign-pattern of the Jacobian, corresponding to the case of only positive feedback in the system (such as Leontief input-output systems, Carvalho and Tahbaz-Salehi, 2019; McKenzie, 1960).

The bounds help solve the aforementioned comparative static challenges because they do not require a matrix inversion of the Jacobian, which, following from the implicit function theorem, is a necessary step in the calculation of the *exact* comparative static. The inverse describes the total feedback reverberation of a shock across all nodes (a Leontief inverse in production network models), and therefore depends on all parameters of the model, and in an analytically complicated manner, leading to the challenges. This is especially acute in high dimensional models as the Jacobian matrix is very large and heterogeneous. I show that one can bound the inverse using only partial information about the non-inverted Jacobian, hence, generating low dimensional sufficient statistics for, and circumventing the analytic complexity of, the comparative statics.

I demonstrate how the bounds can be used to solve three difficult problems in the literature, while showcasing very different styles of application. Two exploit the bounds requiring less data, with the object of interest in the first being a structural parameter of the model, and in the second being the comparative static itself. The third exploits the relatively simpler functional form of the bounds.

1) A key challenge in the peer effect literature is that (point) identification of the peer effects parameter typically requires observation of the entire social network, yet this data is usually only partially available at best (Blume et al., 2015; Bramoullé et al., 2020; Lewis and Chandrasekhar, 2011). In the workhorse linear-in-means model, I show that my comparative static bounds can be inverted to provide a lower bound on the peer effect parameter. This lower bound is useful because it can be identified without any data on the social network beyond the number of friends each person

has, which is much more feasible to attain. Moreover, this does not require network formation assumptions to predict the missing links, which is the solution often used in the literature (see e.g. Breza et al., 2020). I demonstrate this using the Add Health dataset and estimate a lower bound of 0.69, which is close to the estimated point value of 0.78, with the latter using data on the entire social network.

2) Trade economists very regularly estimate the welfare gains from trade liberalizations — the comparative static of welfare with respect to trade costs — yet the standard ex-ante sufficient statistic requires knowledge of the global network of international trade (Arkolakis et al., 2012). This data doesn't exist going back more than sixty years, limiting its application in economic history despite the clear interest in doing so (Findlay and O'Rourke, 2007). I show, however, that an ex-ante sufficient statistic for the bounds on the welfare change requires only the import share of GDP, which is much more readily available. I calculate these bounds for the UK over the past 800 years using data from the Bank of England. The bounds are very narrow, being no wider than $\pm 2.5\%$ of the midpoint for all years prior to 1800.

3) Price-cost passthrough is one of the oldest questions in economics (Marshall, 1890). Theoretically characterizing the magnitude has been limited to models of symmetric firms in order to keep the relationship of price to cost tractable (Dixit, 1986; Weyl and Fabinger, 2013).[2] I show that my bounds generalize the established condition (log-convexity of demand) for more than complete passthrough to general asymmetric, many-firm models.

The outline of this paper is as follows. In section 2, I describe the general framework and derive the comparative statics. In section 3, I review diagonal dominance and present the bounds on the comparative statics. In section 4, I provide a step-by-step guide to applying the bounds, and present the applications to problems in the literature. In section 5, I conclude.

Literature. To my knowledge, no such general method exists to bounding both the magnitude and sign of comparative statics in high dimensional models using low dimensional sufficient statistics. There is a large literature that has sought to determine the sign (but not the magnitude) of comparative statics under the most general as-

[2] Highlighting the intractability of inverting a high dimensional Jacobian, Dixit (1986) writes on pg 119 "...the matrix form is useful in clarifying why I did not think it worthwhile to examine oligopoly with a general form of product heterogeneity. ...No structure could be imposed on its inverse, and no meaningful results could emerge."

sumptions, such as the traditional qualitative economics (Bassett et al., 1967; Hale et al., 1999) or monotone comparative statics (Barthel and Sabarwal, 2018; Milgrom and Shannon, 1994; Quah, 2007; Villas-Boas, 1997).[3] The appeal there, as in the present paper, is that the implications about the comparative statics can be robust to, or agnostic about, specific (quantitative) model assumptions.

The assumption of a diagonally dominant Jacobian has a long history in economics (McKenzie, 1960), often being invoked to guarantee uniqueness or stability of equilibria (Adão et al., 2023; Allen et al., 2020; Dixit, 1986; Gale and Nikaido, 1965; Hadar, 1965; Kolstad and Mathiesen, 1987) or for an invertible demand system (Berry et al., 2013; Cheng, 1985). Diagonal dominance has also been used to characterize the sign of comparative statics in specific frameworks, such as in competitive models (Arrow and Hahn, 1971 theorem T.10.5), oligopoly models (Dixit, 1986), and trade models (Allen et al., 2020; Jones et al., 1993).[4] My results significantly generalize the application of diagonal dominance, being valid in any model for which diagonal dominance is satisfied, and establishes bounds on the magnitude in addition to the sign of the comparative static.

The paper contributes to the vast econometrics literature on partial identification (for surveys, see Kline and Tamer, 2023; Tamer, 2010), particularly to those applications in network models (de Paula and Tang, 2012; de Paula et al., 2018; Miyauchi, 2016). I offer a new method for deriving bounds by exploiting diagonal dominance. By allowing one to avoid inverting a matrix, my approach is related to the literature on weak instruments in econometrics which, to avoid near-zero denominators, relies on the properties of the problem before matrix inversion (see, for example, Horowitz, 2021 or classic works like Anderson and Rubin, 1949).

Low dimensional sufficient statistics have related applications in various strands of the literature. Point identification of comparative statics on welfare using sufficient statistics have been developed using Hulten's theorem (Hulten, 1978, see Baqaee and Rubbo, 2023 for a recent review) and in public finance (Chetty, 2009; Kleven, 2021). Graphical reconstruction methods predict missing links in network models using more readily available low dimensional variables, such as aggregated relational data in social

[3] The Le Chatelier's principle ranks comparative static magnitudes in the short vs the long run (Milgrom, 2006; Dekel et al., 2023).

[4] A wider literature has characterized comparative static signs using diagonal dominance in conjunction with sign restrictions (Carvalho and Tahbaz-Salehi, 2019; Carvalho et al., 2021; McKenzie, 1960; Simon, 1989). I consider this special case in remark 5.

networks (Breza et al., 2020; McCormick and and Zheng, 2015; Sadler, 2025), and balance sheet data in financial networks (Anand et al., 2018; Glasserman and Young, 2016).[5] Sufficient conditions for equilibrium uniqueness using low dimensional sets of parameters have been developed in the international trade and economic geography literatures (Allen et al., 2020, 2024; Kucheryavyy et al., 2023).

The theory developed in this paper applies a result from the linear algebra literature in Ostrowski (1952), who provides bounds on the inverse of diagonally dominant matrices. I innovate on this result by proving that the bounds are sharp, and deriving its implications for comparative statics. The only other apparent usage in economics of the bounds from Ostrowski (1952) is in my earlier work, Norris (2025). This is a much more limited application of Ostrowski (1952) that does not exploit all of the implications, and is only applied to a specific international trade model.

2 Model

Consider a system of $i \in \{1, ..., N\} \equiv \mathcal{N}$ nodes (e.g. agents, countries, products). Each node has an endogenous state, $y_i \in \mathbb{R}$ (e.g. the price of product i), with the system being subject to an exogenous shock, $x \in \mathbb{R}$, (e.g. a demand shifter of one of the products).[6] The state of all nodes are determined jointly by the following equations of state

$$\forall i \in \mathcal{N}: \quad 0 = f_i\left(\boldsymbol{y}, x\right) \tag{1}$$

where $\boldsymbol{y} = \{y_j\}_{j \in \mathcal{N}}$. The function $f_i : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is continuously differentiable, and is typically derived from the equilibrium conditions in the underlying economic model. For example, $f_i\left(\boldsymbol{y}, x\right)$ could be the excess demand for product i, with y_i the price of product i, and x a demand shifter for some product. A solution to equation (1) corresponds to an equilibrium, and I denote this by $\boldsymbol{y}^*\left(x\right)$; the solution needn't be unique. I denote by ∇_{ij} ("nabla") the partial derivative of f_i with respect to endogenous state y_j,

$$\nabla_{ij} \equiv \frac{\partial f_i\left(\boldsymbol{y}, x\right)}{\partial y_j} \tag{2}$$

[5]Galeotti et al. (2024) consider the case where one has noisy measures of the network.

[6]The notation is very general: x could instead represent an aggregate shock to all nodes, or be a scalar parameterizing a shock to a subset of nodes.

and refer to this simply as the Jacobian throughout the paper. The dependence of ∇ on $\{\boldsymbol{y}, x\}$ is suppressed. Intuitively, the Jacobian describes the endogenous feedback between nodes in the system (how the equation of state for node i responds to a change in node j's state). For example, if equation (1) is the system of reduced excess demands in a pure exchange economy, then ∇_{ij} is the cross price effect of demand for product i with respect to price j.

Comparative statics of the system are considered by taking an infinitesimal perturbation in x about a solution $\boldsymbol{y}^* (x)$, and examining the resulting change in state $\boldsymbol{y}$. Only first order effects are considered in this paper. Applying the implicit function theorem to equation (1), the infinitesimal change in the state is given by

$$\frac{\partial y_i}{\partial x} = -\sum_{j \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ij} \frac{\partial f_j}{\partial x} \tag{3}$$

where $\frac{\partial f_j}{\partial x}$ is the partial derivative of f_j with respect to the exogenous shock, x, with its dependence on $\{\boldsymbol{y}, x\}$ suppressed. I refer to $\frac{\partial f_j}{\partial x}$ as the vector of direct effects (equal to the shift in demand in the case where f_i is excess demand and x is a demand shifter of product i). Equation (3) assumes ∇ is invertible, which is guaranteed under the diagonal dominance assumption 1.

The comparative static $\frac{\partial y_i}{\partial x}$ depends on the inverse of the Jacobian, ∇^{-1}. Intuitively, the matrix inverse shows up in the comparative static as it appropriately aggregates all the endogenous feedback in the system from the shock. That is, the effect of the shock x on the state in i incorporates not only the direct effect of x on y_i, but also the indirect effect via the changes in states of all other nodes. States in other nodes $y_{j \neq i}$ respond to a change in state y_i, and this in turn causes y_i to change again. This feedback between nodes is precisely what the Jacobian describes, and the aggregation of all this feedback throughout the system is described by the inverse of the Jacobian.[7]

Two challenges confronting comparative statics arise due to the presence of the matrix inverse. First, a single element of the inverted Jacobian ∇^{-1} depends on all

[7]This can be seen using the standard logic of the Neumann expansion, $\left\{\nabla^{-1}\right\}_{ij} = \frac{1}{\nabla_{jj}} \left\{\left(I - \tilde{\nabla}\right)^{-1}\right\}_{ij} = \frac{1}{\nabla_{jj}} \sum_{k=0}^{\infty} \left\{\tilde{\nabla}^k\right\}_{ij}$, where $\tilde{\nabla}_{ij} \equiv -\frac{\nabla_{ij}}{\nabla_{ii}} (1 - I_{ij})$. The sum converges under assumption 1. See Carvalho and Tahbaz-Salehi (2019) for a discussion in the context of the production network model.

$N \times N$ elements of the non-inverted Jacobian ∇. Consequently, complete identification of the model's parameters is typically required even if one is only interested in the comparative static on a single state. Second, the functional form of this dependence is analytically complicated, being highly non-linear in the parameters of the model (note that ∇ is often linear in the model parameters, for instance equaling the price effects of demand in the aforementioned examples), thus obscuring the economics and properties. In the next section, I derive bounds on the comparative static that do not require inverting the Jacobian. Thus, potentially alleviating these two issues.

3 Theoretical Results

I review diagonal dominance in subsection 3.1. I present my main results on bounding comparative statics in subsection 3.2 and variations in subsection 3.3.

3.1 Diagonal Dominance

The only additional assumption imposed on the Jacobian for the main results is assumption 1: diagonal dominance (in section 3.3, I consider alternative forms of diagonal dominance).[8] $|z|$ denotes the absolute value of $z \in \mathbb{R}$.

Assumption 1. (Diagonal Dominance). *At $\boldsymbol{y}^*(x)$, the Jacobian is strictly column diagonally dominant,*

$$\forall i \in \mathcal{N}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$$

Formally, assumption 1 is referred to as strict column diagonal dominance (see e.g. Horn and Johnson, 2012 definition 6.1.9.); I refer to it simply as diagonal dominance for convenience. Note that assumption 1 is sufficient for ∇ to be non-singular by the Levy–Desplanques theorem (Horn and Johnson, 2012, theorem 6.1.10.a), and thus ∇^{-1} in the comparative static, equation (3), is well-defined under assumption 1. Under the model in section 2, diagonal dominance has the interpretation of the feedback within a node, $|\nabla_{ii}|$, being greater than the feedback between nodes, $\sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$. For example, in the case where equation (1) is the system of excess demands in a

[8]There is an invariance in the system described by equation (1): one can rearrange the order of i and the order of the arguments y_j in $f(\boldsymbol{y}, x)$. Correspondingly, the order of the rows and columns of the Jacobian can be rearranged without loss of generality. Assumption 1 can be applied to any arrangement.

pure exchange economy, assumption 1 implies that the own price effect of demand is greater than the sum of all cross-price effects of demand (all in absolute terms), for each price (Arrow and Hahn, 1971 pg 233).

Diagonal dominance is often invoked in sufficient conditions for equilibrium uniqueness or stability. Local uniqueness is implied because the Jacobian is invertible (Mas-Colell et al., 1995, proposition 17.D.1). Local stability is implied if one also assumes the diagonal elements are all positive, $\forall i : \nabla_{ii} > 0$ (Hahn, 1982 theorem T.1.7c). The combination over a closed rectangular domain implies global uniqueness (Gale and Nikaido, 1965 theorem 4). In table 1, I show that assumption 1 is satisfied in a range of high dimensional, canonical models across economics under assumptions (column three) that are also typically invoked for equilibrium uniqueness or stability (column four). See appendix C for details on these models.

In presenting the bounds, I define the following object that describes the (inverse) intensity of diagonal dominance in the matrix.[9]

Definition 1. (Inverse Diagonally Dominant (iDD) Degree). *For any matrix ∇, the iDD degree of node $i \in \mathcal{N}$ is*

$$\delta_i \equiv \frac{\sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|}{|\nabla_{ii}|} \tag{4}$$

and the maximal iDD degree across all other nodes is

$$\delta_{-i} \equiv \max_{j \in \mathcal{N} \setminus i} \delta_j \tag{5}$$

with ∇ evaluated at $\boldsymbol{y} = \boldsymbol{y}^* (x)$.

Note that $\delta_i \in [0, 1)$ under diagonal dominance of ∇ (assumption 1). When $\forall i :$ $\delta_i = 0$, then ∇ is a diagonal matrix and is therefore maximally diagonally dominant. As any δ_i increases up from 0, the intensity of diagonal dominance diminishes. Hence, δ_i is an inverse measure of the degree of diagonal dominance.

Under the model in section 2, δ_i can be understood as summarizing the exposure of other nodes to endogenous feedback from node i, relative to the feedback within node i. If $\forall i : \delta_i = 0$, the equation of state for a given i, f_i, in equation (1) does not directly depend on any endogenous state except the state in i, y_i. This is true for the

[9] The iDD degree is inversely related to the *diagonally dominant degree*, $|\nabla_{ii}| - \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$, from the linear algebra literature (Liu et al., 2010; Zhao et al., 2013). When applied to a canonical network model, the iDD degree is proportional to the (weighted) *network degree centrality,* see equation (18).

equations of state for all i. Thus, under $\forall i : \delta_i = 0$, there is no endogenous feedback between any nodes in the system (essentially reducing to a one-node model for each node). As δ_i increases, the feedback between nodes increases.

Drawing on a result from Ostrowski (1952), lemma 1 presents the bounds on and signs of elements in the inverse of a diagonally dominant matrix. $\text{sgn}(z)$ is the sign operator, taking values $-1, 0, 1$ if $z < 0$, $z = 0$, $z > 0$, respectively.

Lemma 1. (Bounds on Inverse Diagonally Dominant Matrices*). Suppose* ∇ *satisfies assumption 1, then* $\forall i \in \mathcal{N}, j \in \mathcal{N} \backslash i$

$$\left|\left\{\nabla^{-1}\right\}_{ii}\right| \in \left[\frac{1}{|\nabla_{ii}|}\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{|\nabla_{ii}|}\frac{1}{1-\delta_i\delta_{-i}}\right] \tag{6}$$

$$\left|\left\{\nabla^{-1}\right\}_{ij}\right| \leq \delta_j \left|\left\{\nabla^{-1}\right\}_{ii}\right| \tag{7}$$

$$\text{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right) = \text{sgn}\left(\nabla_{ii}\right) \tag{8}$$

with ∇ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$*. Conditional on* $\{\nabla_{kk}, \delta_k\}_{k\in\mathcal{N}}$*, the bounds in equations (6) and (7) are sharp.*

Proof. Equations (6) and (8): Ostrowski (1952) equation (14). Equation (7): Ostrowski (1952) equation (13). Ostrowski (1952) assumes strict row diagonal dominance, whereas assumption 1 is strict column diagonal dominance. Thus, in applying their bounds, one must replace the matrix with its transpose, notably in their equations (1) and (13). See appendix A.1 for proof of the bounds being sharp. □

The power of lemma 1 is that, despite matrix inverses depending on the entire original matrix, and in a highly complicated manner for general N, the bounds and sign depend only partially on the original matrix, and with a very simple form for all N. Notably, they depend only on the diagonal elements, ∇_{ii}, and the iDD degrees, δ_i. Moreover, conditional on assumption 1 and this information for all nodes, $\{\nabla_{ii}, \delta_i\}_{i\in\mathcal{N}}$, the bounds in lemma 1 are *sharp*. This implies there exists a ∇ with the values $\{\nabla_{ii}, \delta_i\}_{i\in\mathcal{N}}$ such that the bounds of equations (6) and (7) hold with equality. I discuss this property in more detail after presenting theorem 1.

3.2 Comparative Statics Bounds

Using the results of lemma 1, I now present the bounds for the comparative statics in theorem 1. In presenting the bounds, I use one more piece of notation for the absolute sum of the direct effects on all nodes other than i

$$\left|\frac{\partial f_{-i}}{\partial x}\right| \equiv \sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial f_j}{\partial x}\right| \tag{9}$$

Theorem 1. (Comparative Static Bounds under Diagonal Dominance). *Suppose ∇ satisfies assumption 1. If for $i \in \mathcal{N}$,*

$$\left|\frac{\partial f_i}{\partial x}\right| \geq \left|\frac{\partial f_{-i}}{\partial x}\right|\delta_{-i} \tag{10}$$

then the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \left|\frac{\partial f_{-i}}{\partial x}\right|\delta_{-i}}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \left|\frac{\partial f_{-i}}{\partial x}\right|\delta_{-i}}{1-\delta_i\delta_{-i}}\right] \tag{11}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{12}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left[\frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \left|\frac{\partial f_{-i}}{\partial x}\right|\delta_{-i}}{1-\delta_i\delta_{-i}}, \frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \left|\frac{\partial f_{-i}}{\partial x}\right|\delta_{-i}}{1-\delta_i\delta_{-i}}\right] \tag{13}$$

with both ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ evaluated at $\boldsymbol{y} = \boldsymbol{y}^*(x)$. Conditional on $\{\nabla_{jj}, \delta_j\}_{j\in\mathcal{N}}$, $\frac{\partial f_i}{\partial x}$, $\left|\frac{\partial f_{-i}}{\partial x}\right|$, the bounds in equations (11) and (13) are sharp.

Proof. See appendix A.2 for the derivation of equations (11), (12) and (13). See appendix A.1 for proof that equations (11) and (13) are sharp. □

Under diagonal dominance of the Jacobian (assumption 1), theorem 1 gives upper and lower bounds on the the comparative static $\frac{\partial y_i}{\partial x}$. If equation (10) is also satisfied, which implies the direct effect of the shock is greatest on node i (e.g. an exogenous tax is levied mostly on product i), then the sign is also determined (note that the interval in equation 13 includes zero when equation 10 isn't satisfied).[10] The power of

[10]The literature makes analogous restrictions to equation (10) on the direct effects when characterizing the sign of comparative statics. For instance, "binary changes" in Arrow and Hahn (1971) chapter 10 or equation (3) in Simon (1989).

theorem 1 is that, analogous to lemma 1, the bounds and sign depend on only a set of *low dimensional sufficient statistics*, and are analytically much simpler than the exact comparative static.

Low dimensional sufficient statistics. The following five objects are sufficient statistics for the bounds and sign of $\frac{\partial y_i}{\partial x}$: the corresponding diagonal element of the Jacobian, ∇_{ii}, the associated iDD degree and the maximum across all other nodes, δ_i, δ_{-i}, and the direct effect on that node and the sum of direct effects on all other nodes, $\frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$. They are low dimensional because, in comparison, the exact value of $\frac{\partial y_i}{\partial x}$ depends on all $N^2 + N$ elements of $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$. Importantly, as I show in section 4, identifying the sufficient statistics is often substantially easier than identifying $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$.

Sharp bounds. Conditional on assumption 1 and the values $\{\nabla_{kk}, \delta_k\}_{k\in\mathcal{N}}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$,[11] the bounds in theorem 1 are sharp.[12] This implies there exists $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$ consistent with this information such that the bounds of equations (11) and (13) hold with equality. This is a useful property because it reveals the strongest possible logical conclusion about the maximum and minimum values of the comparative static, implied by the given information and model assumptions (notably, diagonal dominance). Another way of saying this is, absent further information on $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$ — such as individual values of $\nabla_{i,j\neq i}$ or $\frac{\partial f_{j\neq i}}{\partial x}$ — and without making stronger assumptions than assumption 1, theorem 1 is the most one can say about the maximum and minimum values of $\frac{\partial y_i}{\partial x}$.[13]

Note that sharp does not necessarily mean narrow. The width of the bounds depend on the values of the sufficient statistics, which in turn depend on the underlying model and parameter values. I discuss the dependence of the width on the sufficient statistics next. I show in section 4 that the width is sufficiently narrow to be useful in a range of applications.

Form of the bounds. Figure 1 visualizes the bounds. The comparative static $\frac{\partial y_i}{\partial x}$ is on the y-axis, and $\delta_i\delta_{-i}$ on the x-axis. Each series corresponds to a different value

[11]The bounds in equation (11) and (13) are enumerated under knowledge of only $\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$; the bounds are the same even if one also has knowledge of $\{\nabla_{kk}, \delta_k\}_{k\in\mathcal{N}\setminus i}$.

[12]The interval is not proved to be sharp; it may contain non-feasible values (Manski, 2003 pg 12).

[13]If one knows more — e.g. some off-diagonal term $\nabla_{i,j\neq i}$ — then narrower bounds may be possible (Kline and Tamer, 2023 pg 130). Section 3.3 explores some alternative information sets and assumptions, though bounds under an arbitrary information set is beyond the scope of this paper.

of $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right|$.[14] The bounds are shown in the case of $-\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = 1$ (multiply all lines by negative one for the case $-\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = -1$).

Start with the special case where the shock only directly affects node i, $\forall k \neq i$: $\frac{\partial f_k}{\partial x} = 0$, which corresponds to the case $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 0$ given by the black-solid line in the figure. This condition implies equation (10) holds, with the bounds on the comparative static magnitude given by equation (11), and the sign given by equation (12), which is positive. Moreover, assume either $\delta_i = 0$, there is no feedback from node i to all other nodes, or $\delta_{-i} = 0$, there is no feedback from other nodes onto node i. Then, the magnitude of the bounds from equation (11) imply point identification of the comparative static magnitude, $\left|\frac{\partial y_i}{\partial x}\right| = \frac{\left|\frac{\partial f_i}{\partial x}\right|}{\left|\nabla_{ii}\right|}$, corresponding to the point on the black-solid line intersecting the y-axis. This is precisely equal to the case of a one-node model, where $f_i\left(\boldsymbol{y}, x\right) = f_i\left(y_i, x\right)$. The comparative static equals the negative of the direct effect $\frac{\partial f_i}{\partial x}$ multiplied by the reciprocal of ∇_{ii} (the matrix inverse reduces to the reciprocal of the diagonal), capturing the within-node feedback on the state from the direct effect. The comparative static sign is clearly given by equation (12).[15]

As the feedback between node i and the other nodes increases from zero, $\delta_i\delta_{-i} > 0$, the comparative static bounds in equation (11) become

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{\left|\frac{\partial f_i}{\partial x}\right|}{\left|\nabla_{ii}\right|}\left[\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{1-\delta_i\delta_{-i}}\right]$$

Intuitively, a "window" around the no feedback case, $\frac{\left|\frac{\partial f_i}{\partial x}\right|}{\left|\nabla_{ii}\right|}$, opens up, with a greater width the greater the feedback, $\delta_i\delta_{-i}$, is. Correspondingly, the upper and lower bounds in figure 1 for the black-solid line diverge as $\delta_i\delta_{-i}$ increase. The window reflects the impact on the comparative static of the between-node feedback, enveloping the impacts that would arise across all permissible configurations of feedback, given by the set of ∇ consistent with the sufficient statistics $\nabla_{ii}, \delta_i, \delta_{-i}$. As the iDD degrees increase, the set of permissible ∇ expands, and the window widens. As a benchmark, at the middle of its domain, $\delta_i\delta_{-i} = 0.5$, the upper (lower) bound is one third above (below) the midpoint of the bounds, as shown by the arrow on figure 1.

In the limit of $\delta_i\delta_{-i} \to 1$, the permissible feedback can be so strong that the upper

[14] $\left|\frac{\partial f_i}{\partial x}\right| \neq 0$ is assumed in figure 1 for cleaner exposition. This is not required for theorem 1.

[15] Intuitively, if $\frac{\partial f_i}{\partial x} > 0$, then, in response to an increase in x, y_i will adjust to bring f_i back down to zero. This requires an increase in y_i, and hence a positive comparative static, if $\nabla_{ii} < 0$.

bound on the magnitude approaches infinity (∇ is possibly singular if $\forall j : \delta_j = 1$). That is, under assumption 1 the maximum amplification in the system arising from the between-node feedback is unbounded. The lower bound, on the other hand, remains finite, with a minimum of $0.5 \cdot \frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}$ at $\delta_i \delta_{-i} = 1$. That is, the maximum attenuation is limited to one half of the value of the no feedback case. Hence, the comparative static sign relative to the no-feedback case cannot be overturned under diagonal dominance, and thus continues to be determined by the same condition, equation (12).

If the shock also directly affects other nodes, $\delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| > 0$, then the bounds across all levels of feedback, $\delta_i \delta_{-i}$, are wider. This reflects the additional impact on the state in i due to the changes in state of other nodes, arising due to these direct effects. The red-dashed and blue-dotted line in figure 1 show this for $\delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right|$ equal to 0.5 and 1 respectively. As long as the direct effects on other nodes are not too strong, i.e. equation (10) holding, then this additional impact on y_i is not enough to cause a change in the sign of the comparative static. The turning point is $\delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 1$ as its lower bound is precisely zero for all values of $\delta_i \delta_{-i}$. As the direct effects on other nodes are increased beyond this point (roughly, the shock is directly incident more on other nodes than it is on i), violating equation (10), the lower bound drops below zero. The green-dot-dash line in figure 1 demonstrates this for $\delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 1.5$. The bounds in equation (13) are used for this case, and the sign can no longer be determined without more information about ∇, $\frac{\partial \boldsymbol{f}}{\partial x}$.

3.3 Variants of the Comparative Static Bounds

Remark 1. **Conditioning on More Information.** Theorem 1 only uses partial knowledge of the vector of direct effects, $\left\{\frac{\partial f_k}{\partial x}\right\}_{k \in \mathcal{N}}$, specifically the direct effect on i, $\frac{\partial f_i}{\partial x}$, and the sum of the direct effects on all other nodes $\left|\frac{\partial f_{-i}}{\partial x}\right|$. In proposition 1 in the appendix, I present the bounds conditional on knowledge of all the direct effects, $\left\{\frac{\partial f_k}{\partial x}\right\}_{k \in \mathcal{N}}$. I apply this result in section C.1.

Remark 2. **Conditioning on Less Information.** Theorem 1 requires knowledge of the maximal degree across of all other nodes, δ_{-i}. In proposition 2 in the appendix, I present the bounds without conditioning on this. I apply this result in section 4.2.

Remark 3. **Row Diagonal Dominance**. Assumption 1 is column diagonal dominance of the Jacobian, ∇. Alternatively, one could assume ∇ is row diagonally dominant: this is equivalent to the transpose of ∇ satisfying assumption 1. I present this result in proposition 3 in the appendix. This result is applied in sections 4.1, 4.3 and C.7.

Remark 4. **Generalized Diagonal Dominance**. The solutions to the equations of state, equation (1), are invariant up to a transformation, $g_i\left[f_i\left(\boldsymbol{y}, x\right)\right] = 0$, for any injective, continuously differentiable function g_i. The Jacobian associated with these transformed equations of state is $g_i'\nabla_{ij}$, where g' is the derivative of g. Assumption 1 applied to this transformed system is $\forall i \in \mathcal{N} : |g_i'|\,|\nabla_{ii}| > \sum_{j\in\mathcal{N}\setminus i}\left|g_j'\right|\,|\nabla_{ji}|$, a condition referred to as generalized diagonal dominance of ∇ (Gao and Wang, 1992). This provides extra scope for applying theorem 1: if ∇ does not satisfy assumption 1, but there exists a $\mathbf{g}'$ such that $g_i'\nabla_{ij}$ does, then theorem 1 can still be applied with $\nabla_{ij} \to g_i'\nabla_{ij}$ (also in definition 1) and $\frac{\partial f_i}{\partial x} \to g_i'\frac{\partial f_i}{\partial x}$. This form of diagonal dominance is considered in e.g. McKenzie (1960) pg 275 and Arrow and Hahn (1971) pg 233. I apply this in section C.2 (equation 103) and discuss this in section C.4 (footnote 38).

I present a notable application in proposition 4, where g_i' is equal to the Perron eigenvector of a transformed version of the Jacobian. ∇ is generalized diagonally dominant if the transformed Jacobian is irreducible and has a spectral radius below one. In models of network games (section C.5), this corresponds to the spectral radius of the adjacency matrix, which is of independent interest (Golub, 2025).

Remark 5. **Signed Diagonal Dominance**. Theorem 1 does not make any assumption on the sign of ∇. If in addition to assumption 1, one assumes all the diagonal elements have the same sign, and all the off-diagonal elements have the opposite sign, with an analogous assumption for the direct effects, then one gets narrower bounds. I present this in proposition 5 in the appendix. This condition corresponds to only positive feedback between nodes, with ∇ (or $-\nabla$, if the diagonal is negative) becoming an M-matrix (Johnson, 1982; Horn and Johnson, 1991, chapter 2.5).

Two workhorse models this includes are Leontief input-output systems (e.g. McKenzie, 1960 theorem 4; Simon, 1989; Carvalho et al., 2021) and, with slight adjustment, competitive models under gross substitutes (e.g. Arrow and Hahn, 1971 chapters 9-10).[16] In each, all diagonal elements of the Jacobian are positive, all off-diagonals are

[16]Gross substitutes satisfies the sign pattern, but is row, generalized diagonally dominant (see Arrow and Hahn, 1971 pg 233-234), rather than column diagonally dominant (assumption 1). One

nonpositive, and the Jacobian is diagonally dominant. These papers have correspondingly characterized the sign of the comparative statics under these conditions. My results build on this by characterizing the comparative static sign without restricting the sign of the Jacobian ∇ (theorem 1), and bounding the magnitude, with or without restrictions on the Jacobian sign (proposition 5 and theorem 1, respectively).[17]

I apply proposition 5 to the peer effect model under pure strategic complements (section 4.1), the baseline production network model (section C.1), and the New Keynesian model under no inferior income effects (section C.2).

4 Applications

I provide a step-by-step guide of applying the bounds in section 4.1, using the ubiquitous linear-in-means model as an example. In the remainder of section 4.1, and in sections 4.2 and 4.3, I detail how the bounds can be used to solve three difficult problems in the literature. In appendix section C, I derive the comparative static bounds in all the models listed in table 1, demonstrating its broad applicability in high dimensional models (many heterogeneous nodes with heterogeneous interactions) across the economics discipline.

4.1 Step-by-Step Guide using the Linear-in-Means Model

The workhorse linear-in-means model is used in many settings analyzing how networks shape economic outcomes or individual decision making (Bramoullé et al., 2016). A prevalent application is on peer effects and will be the focus of this section, though the results generalize. The linear-in-means model is[18]

$$y_i = \beta x_i + \phi \sum_{j \in \mathcal{N}} G_{ij} y_j \tag{14}$$

where y_i is the outcome of individual $i \in \mathcal{N}$ (e.g. the GPA of student i), x_i is an exogenous characteristic (e.g. student gender) and $\beta \in \mathbb{R}$ the direct effect of

could derive an analogue of proposition 5 that applies to this form of signed diagonal dominance, by combining the insights of propositions 3 (row), 4 (generalized) and 5 (signed).

[17] Arrow and Hahn (1971) theorem T.10.5 characterizes the comparative static sign under row, generalized diagonal dominance, without gross substitutes.

[18] Appendix C.5 provides more details, a microfoundation using network games, and includes contextual peer effects.

the characteristic on the outcome. $G_{ij} \in [0,1]$ is the weighted adjacency matrix whose magnitude indicates the strength of the interaction between agents i and j. The adjacency matrix is directed (G is not necessarily symmetric); row-normalized $\forall i \in \mathcal{N} : \sum_{j \in \mathcal{N}} G_{ij} = 1$, so that $\sum_{j \in \mathcal{N}} G_{ij} y_j$ is an average;[19] and there is no self-interaction $\forall i \in \mathcal{N} : G_{ii} \equiv 0$ by convention. $\phi \in \mathbb{R}$ is the peer effect parameter: the causal effect of i's peers' outcomes $\sum_{j \in \mathcal{N}} G_{ij} y_j$ on i's outcome y_i. $\phi > (<)\, 0$ implies strategic complements (substitutes).

Step 1: Map the notation. *Define the comparative static of interest and identify the nodes i, endogenous states y_i, and exogenous shock(s) x.*
Suppose we are interested in the comparative static of individual outcome y_i with respect to the individual characteristics x_j. Then, the nodes are individuals $i \in \mathcal{N}$, the endogenous states are individual outcomes y_i, and the exogenous shocks are individual characteristics x_j. This is a case where we have N potential shocks.

Step 2: Equations of State. *Derive equation (1).*
A minor rearrangement of the linear-in-means model equation (14) gives

$$0 = f_i\left(\boldsymbol{y}, \boldsymbol{x}\right) \equiv y_i - \beta x_i - \phi \sum_{j \in \mathcal{N}} G_{ij} y_j \tag{15}$$

Step 3: Jacobian and Direct Effects. *Derive ∇ (equation 2) and $\frac{\partial f_j}{\partial x}$.*
The Jacobian is equal to the derivative of the equations of state, equation (15), with respect to the individual outcomes,

$$\nabla_{ij} \equiv \frac{\partial f_i}{\partial y_j} = I_{ij} - \phi G_{ij}$$

Feedback in this model, as described the Jacobian, corresponds to the strength of the peer effects between individuals as determined by the peer effect ϕ parameter and the adjacency matrix G_{ij}. The direct effects are equal to the derivatives of the equations of state with respect to the individual characteristics

$$\frac{\partial f_i}{\partial x_j} = -\beta I_{ij}$$

[19] For convenience, I abstract from the possibility of isolated individuals (an individual that has no peers). However, the presented bounds remain valid for isolated individuals.

The comparative static, equation (3), in this model is

$$\frac{\partial y_i}{\partial x_j} = -\sum_{k\in\mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial x_j} = \beta \left\{(I-\phi G)^{-1}\right\}_{ij} \tag{16}$$

Step 4: Diagonal Dominance. *Determine if and when ∇ satisfies assumption 1.*
The Jacobian ∇ in this model satisfies assumption 1 in this model iff $\forall i \in \mathcal{N}$

$$\begin{aligned} |\nabla_{ii}| &> \sum_{j\in\mathcal{N}\setminus i} |\nabla_{ji}| \\ 1 &> |\phi|\, g_i \end{aligned} \tag{17}$$

i.e. the product of the (magnitude of the) peer effect parameter $|\phi|$ and the network in-degree $g_i \equiv \sum_{j\in\mathcal{N}\setminus i} G_{ji}$ (a function of the number of individuals reporting that i is their peer) has to be less than one. This is satisfied if the feedback in the system due to peer effects is not too large, i.e. $|\phi|$ or g_i is sufficiently small.

Step 5: iDD Degree and Sum of Direct Effects on Other Nodes. *Derive δ_i (definition 1) and $\left|\frac{\partial f_{-i}}{\partial x}\right|$ (equation 9).*

$$\delta_i \equiv \frac{\sum_{j\in\mathcal{N}\setminus i} |\nabla_{ji}|}{|\nabla_{ii}|} = |\phi| \sum_{j\in\mathcal{N}\setminus i} G_{ji} = |\phi|\, g_i \tag{18}$$

The iDD degree for individual i is equal to the absolute peer effect parameter $|\phi|$ multiplied by their in-degree g_i, reflecting the amount of feedback in the system. Note that $\delta_i = |\phi|\, g_i \in [0,1)$ under equation (17). The maximal iDD degree across all other nodes is $\delta_{-i} = \max_{j\in\mathcal{N}\setminus i} |\phi|\, g_j \equiv |\phi|\, g_{-i}$. The direct effects on other nodes is

$$\left|\frac{\partial f_{-i}}{\partial x_j}\right| \equiv \sum_{k\in\mathcal{N}\setminus i} \left|\frac{\partial f_k}{\partial x_j}\right| = \sum_{k\in\mathcal{N}\setminus i} |\beta I_{kj}| = |\beta|\,(1-I_{ij})$$

Step 6: Apply the Bounds from Theorem 1. *Check for which nodes equation (10) holds, and apply the bounds from equations (11), (12) and (13) accordingly.*
Assume equation (17) holds so that assumption 1 is satisfied. For the comparative

statics, $\frac{\partial y_i}{\partial x_j}$, equation (10) of theorem 1 holds when $j = i$, as

$$\left|\frac{\partial f_i}{\partial x_i}\right| = |\beta| \geq 0 = \delta_{-i}\left|\frac{\partial f_{-i}}{\partial x_i}\right|$$

Thus, for $j = i$, $\frac{\partial y_i}{\partial x_i}$, bounds on the magnitude are given by equation (11)

$$\left|\frac{\partial y_i}{\partial x_i}\right| \in \left[\frac{|\beta|}{1+\phi^2 g_i g_{-i}}, \frac{|\beta|}{1-\phi^2 g_i g_{-i}}\right] \tag{19}$$

and its sign by equation (12)

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x_i}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x_i}\right) = -\cdot 1 \cdot \operatorname{sgn}\left(-\beta\right) = \operatorname{sgn}\left(\beta\right) \tag{20}$$

For $j \neq i$, $\frac{\partial y_i}{\partial x_j}$, bounds are given by equation (13)

$$\frac{\partial y_i}{\partial x_j} \in \left[\frac{-\left|\phi\beta\right| g_{-i}}{1-\phi^2 g_i g_{-i}}, \frac{\left|\phi\beta\right| g_{-i}}{1-\phi^2 g_i g_{-i}}\right] \tag{21}$$

but the sign is not determined (note that the bounds in equation 21 include zero).

Step 7: Variants on the Bounds. *Consider the variants on theorem 1 given in section 3.3, especially if assumption 1 doesn't hold.*
Assumption 1 for this model, equation (17), is restrictive if there are particularly central individuals in the network: i such that many other individuals report them as peers, leading to a high g_i. The row diagonal dominance variant (remark 3), however, is much less restrictive.[20] Row diagonal dominance, equation (58), is satisfied if

$$\begin{aligned} |\nabla_{ii}| &> \sum_{j\in\mathcal{N}\setminus i} |\nabla_{ij}| \\ 1 &> |\phi| \end{aligned} \tag{22}$$

using $\sum_{j\in\mathcal{N}\setminus i} G_{ij} = 1$ (the out-degree is equal to one) because G is row normalized. Equation (22) is often invoked in linear-in-means models as it is sufficient for a unique and stable equilibrium (see Bramoullé et al. 2016 section 5.4.1). Using proposition 3,

[20] In appendix C.5, I also apply the variant of remark 4.

the iDD degree is $\forall i \in \mathcal{N} : \delta_i = |\phi|$ and the comparative static bounds are

$$\forall i \in \mathcal{N}, j \in \mathcal{N} \backslash i : \quad \left|\frac{\partial y_i}{\partial x_i}\right| \in \left[\frac{|\beta|}{1+\phi^2}, \frac{|\beta|}{1-\phi^2}\right], \quad \left|\frac{\partial y_j}{\partial x_i}\right| \leq |\phi| \left|\frac{\partial y_i}{\partial x_i}\right| \tag{23}$$

with the comparative static sign continuing to be given by equation (20). If one further assumes strategic complements, $\phi \geq 0$, which is the more common case empirically (e.g. in the broad class of settings with social multipliers, Glaeser et al. 2003, such as in education, Calvó-Armengol et al. 2009), then the Jacobian becomes an M-matrix (see remark 5) and the lower bound in equation (23) can be strengthened giving

$$\forall i \in \mathcal{N} : \quad \left|\frac{\partial y_i}{\partial x_i}\right| \in \left[|\beta|, \frac{|\beta|}{1-\phi^2}\right] \tag{24}$$

Step 8: Low Dimensional Sufficient Statistics, Sharpness and Width of the Bounds. Theorem 1 implies that conditional on the low dimensional sufficient statistics, $\{\nabla_{jj}, \delta_j, \}_{j \in \mathcal{N}} \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$, which correspond to the parameter set $|\phi|, \{g_j\}_{j \in \mathcal{N}}, \beta$ in this model, the bounds on $\frac{\partial y_i}{\partial x_j}$ in equations (19) and (21) are sharp. That is, if one only knew the values of $|\phi|, \{g_i\}_{i \in \mathcal{N}}, \beta$ and nothing else, notably no knowledge of the adjacency matrix G_{ij} beyond the in-degrees $\{g_i\}_{i \in \mathcal{N}}$, equations (19) and (21) are the most one can say about the maximum and minimum values the comparative static, $\frac{\partial y_i}{\partial x_j}$, can take.

If one conditions on a different information set, the associated sharp bounds will also be different in general. This is illustrated by equation (23), which conditions on $|\phi|, \beta$, the out-degrees (which all equal one) and the assumption given in equation (22), as opposed to equation (19), which conditions on $|\phi|, \beta$, the in-degrees $\{g_i\}_{i \in \mathcal{N}}$, and the assumption given in equation (17).

The width of the bounds depend on the values of the low dimensional sufficient statistics. Consider equation (23): these bounds are sharp for all values of the low dimensional sufficient statistic $|\phi| < 1$, however their width depends on the value of $|\phi|$, being wider when $|\phi|$ is larger. For example, the bounds are one third above and below the midpoint when $|\phi| = \sqrt{0.5} \approx 0.71$ (see figure 1 noting that $\delta_i \delta_{-i} = |\phi|^2$).

Step 9: Application of the Bounds. *Use the bounds to aid in settings of incomplete knowledge of model parameters, or characterization of comparative static properties.*

The advantage of the bounds is that, compared to the exact comparative static, equation (16), they depend on fewer model parameters (notably G), and have a much simpler functional form (avoiding the inversion a potentially large matrix, $I - \phi G$).

I show how this can be used when interest is in a structural parameter of the model. In this case, the peer effect parameter, ϕ, about which there is considerable attention in the literature (see Bramoullé et al. 2020 for a recent survey).[21]

A key challenge in this literature is that point identification of ϕ requires complete knowledge of G in general, while one typically only partially observes G (Blume et al., 2015; Lewis and Chandrasekhar, 2011). An increasingly large number of recent papers have developed methods to restore point identification by imposing additional assumptions on the data-generating process of G.[22] I show that one can alternatively use my bounds to partially identify ϕ given only partial knowledge of G, without needing to resort to additional assumptions on G. In particular, no *bilateral* network data is required; the number of peers each individual has is sufficient.[23]

To proceed, assume $\phi \in [0, 1)$ and invert the relationship in equation (24) to bound ϕ conditional on $\frac{\partial y_i}{\partial x_i}$. This is useful because we often more readily have information on $\frac{\partial y_i}{\partial x_i}$, which is identified from exogenous variation in an individual characteristic x. Inverting the upper bound in equation (24) for a subset of individuals $\mathcal{N}_1 \subset \mathcal{N}$ yields

$$i \in \mathcal{N}_1 : \quad \phi \geq \sqrt{1 - \left| \beta \Big/ \frac{\partial y_i}{\partial x_i} \right|}$$

β can then be substituted out using the lower bound in equation (24) for a different subset of individuals $\mathcal{N}_2 \subset \mathcal{N}$

$$i \in \mathcal{N}_1, j \in \mathcal{N}_2 : \quad \phi \geq \sqrt{1 - \left| \frac{\partial y_j}{\partial x_j} \Big/ \frac{\partial y_i}{\partial x_i} \right|} \tag{25}$$

Because this bound holds for all i, j within each subset, and because the comparative

[21] I discuss in section C.7 how one could analogously apply this method to identify the spatial lag parameters in spatial econometric models.

[22] For example, by assuming sparsity (Blume et al., 2015), by parameterizing the network formation process (Auerbach, 2022; Breza et al., 2020; Lewis and Chandrasekhar, 2011) in order to implement graphical reconstruction, or by assuming that one observes many instances of the network (de Paula et al., 2024; Lewbel et al., 2023).

[23] This is reminiscent of using aggregated relational data for identification without bilateral network data, however those methods also require assumptions on the network formation process underlying G (Breza et al., 2020; McCormick and and Zheng, 2015).

statics all have the same sign (by equation 20), then equation (25) holds for averages of the comparative statics within each subset of individuals. In particular,

$$\phi \geq \sqrt{1 - \left|\frac{b_2}{b_1}\right|} \tag{26}$$

where b_k is estimated from the following regression for $k \in \{1, 2\}$

$$i \in \mathcal{N}_k : \quad y_i = b_k x_i + \varepsilon_{k,i} \tag{27}$$

assuming x_i is exogenous, i.i.d. and mean zero, and $\varepsilon_{k,i}$ is the residual.[24] The intuition for equation (26) is as follows. Two groups of individuals can have different comparative statics, and therefore b_k, only because of heterogeneous exposure to peer effects, arising from being in different positions in the network G (specifically, $\left\{(I - \phi G)^{-1}\right\}_{ii}$ in equation 16). My theory shows that the range of values that the comparative static can take across any position in the network is determined by ϕ — equation (24). Hence, one can infer how large ϕ must at least be in order to rationalize any observed difference in the comparative statics, $\left|\frac{b_2}{b_1}\right|$, which is equation (26).

Equation (26) is decreasing in $\left|\frac{b_2}{b_1}\right|$, so choosing two subsets $\mathcal{N}_1, \mathcal{N}_2$ that lead to greater differences in the comparative statics will be more informative on ϕ. Limited information on G is sufficient for this choice. For example, using the number of peers an individual i has, n_i, as the comparative static is likely quite different for individuals with few vs many friends.[25] Note that even an imperfect measure of n_i (such as due to censoring) is viable as the bounds are valid under any choice of subsets $\mathcal{N}_1, \mathcal{N}_2$.

This provides a method of partial identifying ϕ with very limited information on G, while point identification of ϕ requires complete knowledge of the entire bilateral

[24]The regression coefficient identifies $b_k = \frac{\sum_{i \in \mathcal{N}_k} E[y_i x_i]}{\sum_{i \in \mathcal{N}_k} E[x_i^2]} = \frac{1}{|\mathcal{N}_k| \sigma_x^2} \sum_{i \in \mathcal{N}} E\left[\sum_{k \in \mathcal{N}_k} \frac{\partial y_i}{\partial x_k} x_k x_i\right] = \frac{1}{|\mathcal{N}_k|} \sum_{i \in \mathcal{N}_k} E\left[\frac{\partial y_i}{\partial x_i}\right]$ where σ_x^2 is the variance of x (exogenous stochastic G is permitted). The second equality used that the solution y_i of equation (14) is linear in $\boldsymbol{x}$ and the third used that x_i is i.i.d. Because all $\frac{\partial y_i}{\partial x_i}$ have the same sign, then $|b_k| = \frac{1}{|\mathcal{N}_k|} \sum_{i \in \mathcal{N}_k} E\left[\left|\frac{\partial y_i}{\partial x_i}\right|\right]$.

[25]The dependence in the simple case where everyone is connected to each other within their subset with equal intensity, and to no-one outside, is $\left\{(I - \phi G)^{-1}\right\}_{ii} = \frac{n_i + \frac{\phi}{1-\phi}}{n_i + \phi}$.

network G. The latter is usually implemented using the following regression

$$y_i = \beta x_i + \phi \sum_{j \in \mathcal{N}} G_{ij} y_j + \tilde{\varepsilon}_i \tag{28}$$

with $\sum_{j \in \mathcal{N}} G_{ij} y_j$ instrumented by $\sum_{j \in \mathcal{N}} G_{ij} x_j$ (Bramoullé et al., 2009). These terms can be constructed, and therefore the estimation is feasible, only if G is fully known.

I illustrate this using the Add Health dataset (https://addhealth.cpc.unc.edu), which has been used abundantly for the analysis of peer effects (see Calvó-Armengol et al. 2009 for early work). The data comes from surveys of adolescents who were in grades 7-12 during the 1994-95 school year. Variables include student characteristics, school performance, and partial data on friendship networks (identities of the top ten friends). I consider y_i to be student i's GPA, and x_i to be gender (equal to one if female, zero otherwise). Thus, β is the direct effect of a student's gender on their GPA, and ϕ the causal effect on their GPA of their friends' GPA.

The results are presented in table 2. In all columns I include a list of control variables as in Bramoullé et al. (2009) (up to the limitations of the public-use sample). I choose the subsets based on the number of friends reported by each student i, n_i, with $\mathcal{N}_1 : n_i \leq 4$ and $\mathcal{N}_2 : n_i \geq 5$. Columns (1) and (2) are the regressions in equation (27) for each subset. This yields $\left|b_2/b_1\right| = 0.53$, giving $\phi \geq 0.69$ using equation (26). For comparison, column (3) is the regression in equation (28) using the observable G, giving a point estimate of $\phi = 0.78$. Thus, the bound on ϕ, which uses data on only the number of friends, is quite close to the point estimate of ϕ, which uses data on the entire observable friendship network. Moreover, the latter may be biased as G is only partially observed (censored without relationship intensity) while the bounds, recall, are still valid, even though the implied n_i may be mis-measured.

4.2 Gains from Trade Liberalizations

A primary question in the International Trade literature is what are the welfare gains from reducing international trade costs? This is often operationalized using a quantitative trade model to calculate the comparative static of country i (the nodes) welfare W_i (the endogenous state) to a change in trade costs $\overline{\tau}$ (the exogenous shock). The seminal paper Arkolakis et al. (2012) proved that, in many standard trade models, this can be calculated from knowledge of only the own-import share X_{ii}/Y_i before and after the trade cost shock, and the trade elasticity $\phi > 0$ (often equal to one plus the cross-country elasticity of substitution). Specifically,[26]

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -\frac{1}{\phi} \frac{\partial \ln \left(X_{ii}/Y_i\right)}{\partial \ln \overline{\tau}} \tag{29}$$

where X_{ij} is the trade flow from country i to j and $Y_i \equiv \sum_{j \in \mathcal{N}} X_{ij}$ is GDP. An important limitation of equation (29) is that it is an *ex-post* sufficient statistic: knowledge of X_{ii}/Y_i after the shock is required. One cannot in general simply use the observed X_{ii}/Y_i after the trade cost $\overline{\tau}$ change because there may be confounding shocks (say, from concurrent changes in productivity). One also cannot apply equation (29) to counterfactual liberalizations as the own-import share after is not observable.[27] That is, equation (29) can only be applied to observable, exogenous changes in $\overline{\tau}$.

The alternative approach that does not require post-shock information — an *ex-ante* sufficient statistic — instead requires knowledge of the full matrix of bilateral trade flows before the shock $\{X_{ij}\}_{\{i,j\} \in \mathcal{N}^2}$ (proposition 2 vs 1 in Arkolakis et al., 2012). Although this has the advantage of being applicable to endogenous or unobservable $\overline{\tau}$, it is infeasible when the full trade flow matrix is not known because of data limitations.

I show that the ex-ante sufficient statistic requires only X_{ii}/Y_i before the shock (and ϕ) for the *bounds* on the welfare change, in the case where $\overline{\tau}$ is a proportional increase in country i trade costs with the rest of the world.[28] This is a significantly reduced data requirement; moreover, with the bounds being sharp, this is the most one can say about the welfare change without more data on the trade matrix. I require an additional assumption, though, relative to the aforementioned ex-ante sufficient statistic, of quasi-symmetric trade costs (symmetric up to an origin and destination shifter), but this is a fairly common modeling restriction (Allen et al., 2020).

Under these assumptions, the ex-ante sufficient statistic for the point value of the welfare change is (see appendix C.3 for the full details and derivation)

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -2\frac{1+2\phi}{1+\phi} \left\{\nabla^{-1}\right\}_{ii} \left(1 - \frac{X_{ii}}{Y_i}\right), \quad \nabla_{ij} = I_{ij} + \frac{\phi}{1+\phi}\frac{X_{ij}}{Y_i} \tag{30}$$

[26]Equation (29) holds with arbitrary changes in $\overline{\tau}$, as noted in Arkolakis et al. (2012). I present the differential form to allow cleaner comparison to my bounds, which hold only in differential form.

[27]Except in the special case where the country is in autarky after the trade cost change. Then, the change in welfare is equal to $\ln\left(X_{ii}/Y_i\right)^{1/\phi}$ (see Arkolakis et al., 2012 corollary 1).

[28]See equation (121) in the online appendix for the general case.

As noted above, the full trade matrix $\{X_{ij}\}$ before the shock is required. The Jacobian ∇ in this model captures feedback between countries (the nodes) that arises due to demand substitution: if prices fall in one country, other countries consume more from them. This feedback is stronger if the trade elasticity ϕ is higher, and if countries buy more from each other, $\frac{X_{ij}}{Y_i}$.

Assumption 1 is satisfied and one can apply the results of proposition 2 (so that knowledge of δ_{-i}, and therefore $\forall j \in \mathcal{N}\backslash i : \frac{X_{jj}}{Y_j}$, is not required, see remark 2)

$$-\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} \in \frac{2\phi+1}{\phi} 2\delta_i \left(\frac{1}{1+\delta_i}, \frac{1}{1-\delta_i} \right), \quad \delta_i = \frac{\phi}{1+\phi} \frac{1 - \frac{X_{ii}}{Y_i}}{1 + \frac{\phi}{1+\phi}\frac{X_{ii}}{Y_i}} \tag{31}$$

The bounds on the welfare change given in equation (31) only require knowledge of $\frac{X_{ii}}{Y_i}$ before the shock; $\frac{X_{ii}}{Y_i}$ after the shock is not needed (in contrast to equation 29), and the full trade flow matrix is not needed (in contrast to equation 30).

I apply this in a prominent setting where the data limitation is binding: economic history. Data on the full trade matrix is only available after 1962 (with the introduction of the Comtrade dataset), yet the gains from trade is still a question of interest before this period (Federico and Tena, 1991; Findlay and O'Rourke, 2007). Nonetheless, we do have data on total imports, $IM_i \equiv \sum_{j \in \mathcal{N}\backslash i} X_{ji}$, and GDP (and therefore X_{ii}) for some countries much further back in time (Federico and Tena-Junguito, 2017; Müller et al., 2025). Thus, partial identification of $\frac{\partial \ln W_i}{\partial \ln \overline{\tau}}$ using equation (31) is feasible, while point identification using equation (30) is impossible.

I calculate the bounds on $\frac{\partial \ln W_i}{\partial \ln \overline{\tau}}$ for the UK over the past 800 years using data from the Bank of England ("A millennium of macroeconomic data", https://www.bankofengland.co.uk/statistics/research-datasets). I set $\phi = 8$ following Jacks et al. (2011); Mitchener et al. (2022). Figure 2a presents the results. The bounds are wider in more recent times, with a width of $\pm 15\%$ relative to the midpoint from the year 2000 onwards, and much narrower further back in time, being less than $\pm 2.5\%$ for all years before 1800. This is useful given it is precisely further back in time when we do not have the bilateral trade data and so may want to rely on the bounds for identification.

The reason why the bounds are wider in more recent times is because the value of the iDD degree δ_i for $i = UK$ is greater (figure 2b). δ_i is an increasing function of the import share, $\frac{IM_i}{Y_i} = 1 - \frac{X_{ii}}{Y_i}$ (see equation 31), and the latter has been increasing in recent times (figure 2b). The gains from trade depend on the exact pattern of trade between countries. If a country has a larger import share $\frac{IM_i}{Y_i}$, there are more permissible patterns of bilateral trade flows, $\{X_{ij}\}$, leading to a wider range of values for the gains from trade. Hence, causing the bounds to be wider.

4.3 Cost-Price Passthrough

There has been longstanding interest in the literature concerning the passthrough of cost shocks to prices. This dates back to Marshall (1890) in the context of tax incidence, with more recent work leveraging passthrough as a sufficient statistic for various welfare analyses (Chetty, 2009). In imperfectly competitive settings, one property that has been explored is when passthrough is more than complete, which implies a number of qualitative properties (Anderson et al., 2001; Bagnoli and Bergstrom, 2005; Stern, 1987; Weyl and Fabinger, 2013). However, theoretical characterizations are typically limited to symmetric models, as the dimensionality of the problem is reduced, permitting a tractable analysis (see Dixit, 1986 pg 119). Under symmetric imperfect competition and linear cost, passthrough exceeds unity iff demand is log-convex (Bulow and Pfleiderer, 1983; Seade, 1985; Weyl and Fabinger, 2013). Using the results of the current paper, I offer a generalization to asymmetric models.

Consider a Bertrand oligopoly with differentiated products. There are $i \in \mathcal{N}$ firms (the nodes) each producing one product with constant marginal cost, c_i (exogenous shocks), and with twice-continuously differentiable demand $D_i(\boldsymbol{p})$, where $\boldsymbol{p} = \{p_i\}_{i\in\mathcal{N}}$ are the product prices (endogenous states). The log profits of firm i are

$$\pi_i(\boldsymbol{p}, c_i) = \ln\left[(p_i - c_i) D_i(\boldsymbol{p})\right]$$

and the firm chooses p_i to maximize $\pi_i(\boldsymbol{p}, c_i)$, taking all other firms' prices as given. The passthrough to p_i of a change in c_i is given by (see appendix C.4 for details)

$$\frac{\partial p_i}{\partial c_i} = \left\{\nabla^{-1}\right\}_{ii}, \qquad \nabla_{ij} = \frac{c_i}{p_i} I_{ij} - \left(1 - \frac{c_i}{p_i}\right)^2 \frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j} \tag{32}$$

Passthrough is greater than one, $\frac{\partial p_i}{\partial c_i} > 1$, iff $\{\nabla^{-1}\}_{ii} > 1$. For models with many firms (large N), $\{\nabla^{-1}\}_{ii}$ is generally a very complicated object, thus determining under what conditions it's greater than one is challenging. The literature typically imposes complete symmetry in demand and costs across all firms to make this tractable. The results of section 3 allow one to proceed without resorting to symmetry.

Recalling that log-convexity of demand is the relevant condition in the symmetric case, a sufficient condition for demand to be log-convex in the general case is when the Hessian of log demand, $\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j}$, is row diagonally dominant with positive diagonal. Assuming the convexity isn't too strong so that the equilibrium is stable, this condition is also sufficient for ∇ to be row diagonally dominant, hence one can use the bounds from proposition 3 (see remark 3). The resulting lower bound is greater than one, implying passthrough is always greater than unity under this condition. Thus, log-convexity of demand is tightly related to more than complete passthrough in asymmetric models, just like in symmetric models.

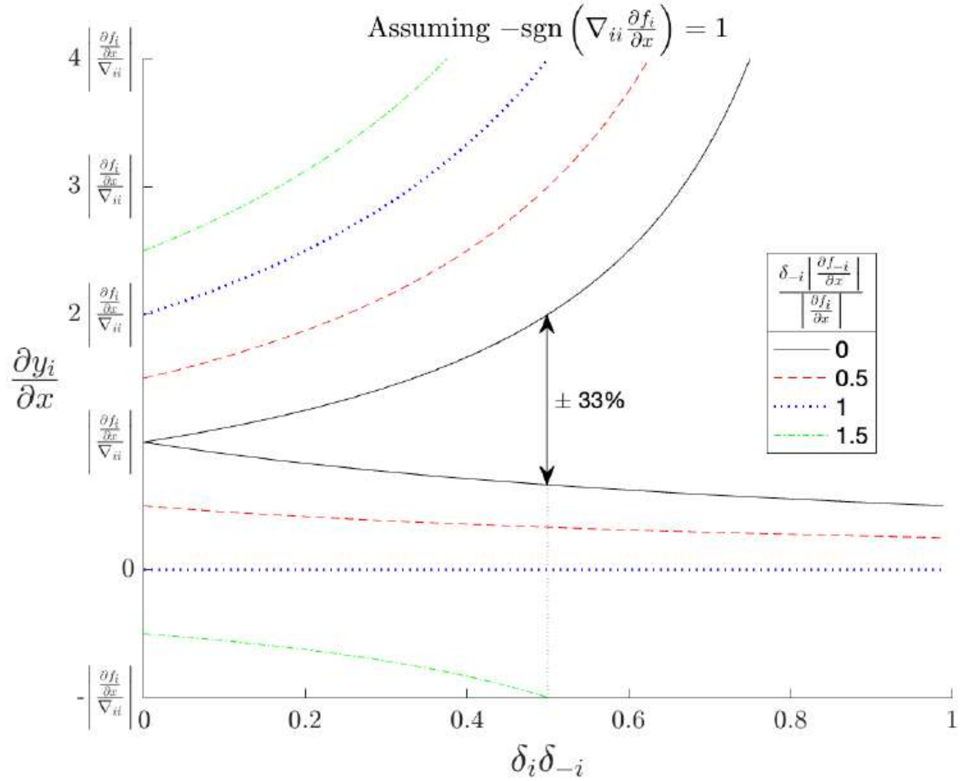
5 Conclusion

In this paper I revisit an old inquiry in economics: what can we deduce about comparative statics while making as few assumptions as possible? (Bassett et al., 1967; Hale et al., 1999) I offer new results advancing the frontier in this subject. I exploit the widely-used assumption of diagonal dominance of the Jacobian, and show this implies novel bounds on comparative statics.

The value of this result is twofold. First, the bounds are identified using low dimensional sufficient statistics. This permits one to still learn about the comparative static in cases where full knowledge of the model parameters is infeasible. Because the bounds are sharp, they are the most one can say about the maximum and minimum values the comparative static can take, if the sufficient statistics are all that is known. Moreover, if direct knowledge of the comparative statics is available instead, the methodology can be inverted to bound structural parameters of the model.

Second, they are analytically simpler than the exact relationship. This potentially allows for easier theoretical characterization of the comparative statics, offering new insight into the underlying economic mechanisms. This provides an alternative to resorting to restrictive, stylized, or low-dimensional versions of the model.

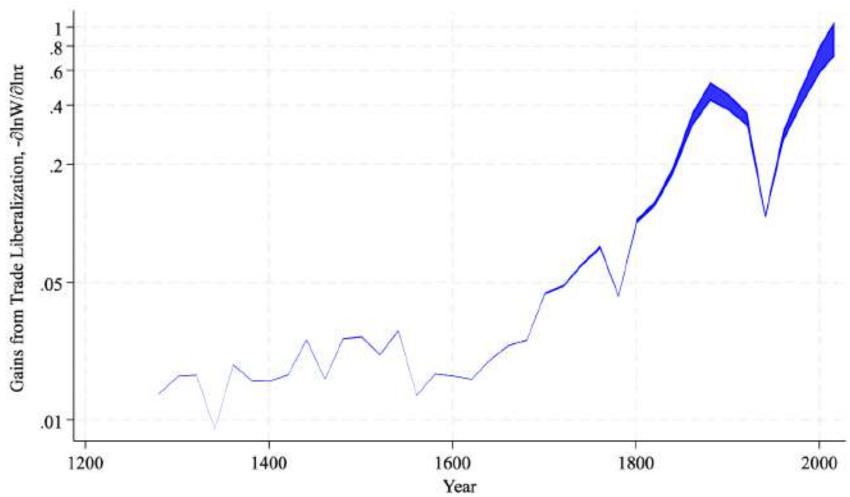
Figure 1: Visualization of Theorem 1



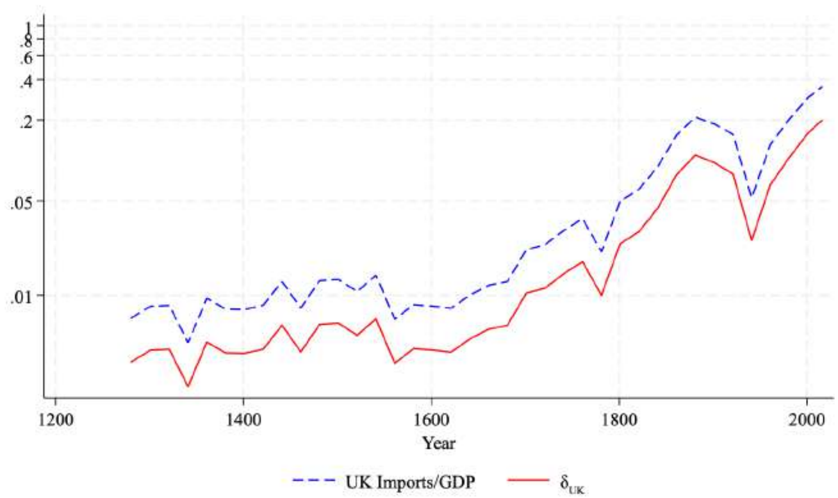
Notes. The bounds on the comparative static from theorem 1 are displayed in the case of $-\text{sgn}(\nabla_{ii})\,\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = 1$. Each series shows the upper and lower bounds (inclusive of the sign) on the comparative static under different values of $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right|$. The arrow labels the width of the bound relative to its midpoint at $\delta_i\delta_{-i} = 0.5$ for the case $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 0$.

Figure 2: The Gains from Trade Liberalization in the UK over 800 Years

(a) Gains from Trade Liberalization



(b) Import share of GDP and the iDD degree, δ_i



Notes. In subfigure (a), the bounds are presented for the elasticity of UK welfare with respect to a proportional reduction in trade costs with the rest of the world. In subfigure (b), the import share of GDP and the iDD degree δ_i are presented. Data is from the Bank of England.

Table 1: Diagonal Dominance in Economic Models

	Field/Model	Reference	Sufficient Condition for Diagonal Dominance	Relevance of Sufficient Condition to Model
Macroeconomics	Production Networks	Carvalho and Tahbaz-Salehi (2019)	Always	—
	New Keynesian	Auclert et al. (2024)	$\forall s, t : \text{MPC}_{st} > 0$ MPC: marginal propensity to consume s, t: time periods	No Inferior Income Effects
Microeconomics	International Trade / Economic Geography	Allen et al. (2020)	$\phi \gtrless -\frac{1}{2},\ \psi \gtrless -\frac{1}{2},\ \tau$ quasi-symmetric ϕ: demand elasticity, ψ: supply elasticity τ: trade cost matrix	Unique Interior Equilibrium
	Industrial Organization: Oligopoly	Milgrom and Roberts (1990)	Demand $\in$ {CES, logit, linear substitutes; restricted translog}	Unique Nash Equilibrium
	Game Theory: Network Games	Bramoullé et al. (2016)	$\lvert\phi\rvert\mu < 1,\ G$ irreducible ϕ: payoff impact parameter G: adjacency matrix μ: spectral radius of $\lvert G\rvert$	Unique Stable Nash Equilibrium
Econometrics	Time Series: ARMA(p,q)	Brockwell and Davis (2016)	$\sum_{s=1}^{p} \lvert\beta_s\rvert < 1$ β_s: s^{th} autoregressive coefficient	Unique Causal Stationary Solution
	Spatial Econometrics	LeSage et al. (2009)	$\lvert\rho\rvert < 1$ ρ: spatial lag parameter	Stable Solution

Notes. Summary of the models that I apply the theoretical results of this paper to (see appendix C for details). Cited is the main reference I use for the model. A sufficient condition for diagonal dominance is presented, either for column (assumption 1), row (remark 3), or generalized (remark 4) diagonal dominance. The last column shows an implication of the sufficient condition that has independent relevance to the model.

Table 2: Bounding Peer Effects without Bilateral Network Data

	(1)	(2)	(3)
Own Gender	0.235***	0.124***	0.166***
	(0.051)	(0.043)	(0.034)
Peers GPA			0.778***
			(0.101)
Observations	1129	1519	2156
Sample	$n_i \leq 4$	$n_i \geq 5$	All
Estimator	OLS	OLS	IV
First-stage F			42.6

Robust standard errors in parentheses.

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Notes. The dependent variable is student GPA. The first row is the student's gender (equal to one if female, zero otherwise). The second row is an average of the student's friends' GPAs, which is instrumented by an average of the student's friends' gender, age and school grade. Each specification also controls for the student's age, school grade, whether white, whether born in USA, whether lives with mother, mother's education, and whether father is present. Column (1) uses the sample of students which report no greater than four friends; column (2) the sample which report at least five friends; column (3) uses the full sample of students. Data is from Add Health.

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A Proofs of Main Results

A.1 Proof of the bounds being sharp in Lemma 1 and Theorem 1

Proof. The bounds are shown to be sharp if, conditional on assumption 1 and the information available ($\{\nabla_{kk}, \delta_k\}_{k\in\mathcal{N}}$ for lemma 1 and $\{\nabla_{kk}, \delta_k\}_{k\in\mathcal{N}}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$ for theorem 1), for each bound there exists a ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ that satisfies the bound with equality. I prove this using the following Jacobian

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} & \nabla_{13} & \cdots & \nabla_{1N} \\ \nabla_{21} & \nabla_{22} & 0 & \cdots & 0 \\ 0 & 0 & \nabla_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \nabla_{NN} \end{pmatrix} \tag{33}$$

i.e. non-zero elements in the second leading principal sub-matrix, the diagonal, and the first row; all other elements are zero. Each non-zero element can be of arbitrary magnitude, conditional on satisfying assumption 1 (i.e. $|\nabla_{21}| \leq |\nabla_{11}|$ and $\forall i \neq 1 : |\nabla_{1i}| \leq |\nabla_{ii}|$), and be of arbitrary sign. The iDD degrees of this ∇ are

$$\delta_i = \begin{cases} i = 1 : & \left|\frac{\nabla_{21}}{\nabla_{11}}\right| \\ i > 1 : & \left|\frac{\nabla_{1i}}{\nabla_{ii}}\right| \end{cases} \tag{34}$$

For the direct effects, I utilize the case where

$$\forall i \notin \{1, 2\} : \quad \frac{\partial f_i}{\partial x} = 0 \tag{35}$$

which implies the direct effect on all nodes other than $i = 1$ to be

$$\left|\frac{\partial f_{-1}}{\partial x}\right| = \left|\frac{\partial f_2}{\partial x}\right| \tag{36}$$

It is sufficient to prove the bounds are sharp for $i = 1$ and with the maximal degree across all other nodes being $\delta_{-1} = \delta_2$. The proof follows equivalently for all other cases under a relabelling of indices in equations (33) and (35).

The only components of this ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ not conditioned on (i.e. are not part of the sufficient statistics) are $\text{sgn}(\nabla_{12}), \text{sgn}(\nabla_{21})$, and also $\text{sgn}\left(\frac{\partial f_2}{\partial x}\right)$ in the case of Theorem 1. I will show that values of these exist such that the bounds hold with equality.

Proof for Lemma 1. The inverse of the matrix of equation (33) is

$$\nabla^{-1} = \begin{pmatrix} \frac{\nabla_{22}}{\phi} & -\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{22}}{\phi} & -\frac{\nabla_{13}}{\nabla_{33}}\frac{\nabla_{22}}{\phi} & \cdots & -\frac{\nabla_{1N}}{\nabla_{NN}}\frac{\nabla_{22}}{\phi} \\ -\frac{\nabla_{21}}{\phi} & \frac{\nabla_{11}}{\phi} & \frac{\nabla_{13}}{\nabla_{33}}\frac{\nabla_{21}}{\phi} & \cdots & \frac{\nabla_{1N}}{\nabla_{NN}}\frac{\nabla_{21}}{\phi} \\ 0 & 0 & \frac{1}{\nabla_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\nabla_{NN}} \end{pmatrix} \tag{37}$$

where $\phi \equiv \nabla_{11}\nabla_{22} - \nabla_{12}\nabla_{21}$. Manipulating the expression for the $(1,1)$ element

$$\begin{aligned} \left\{\nabla^{-1}\right\}_{11} &= \frac{1}{\nabla_{11}}\frac{1}{1-\frac{\nabla_{21}}{\nabla_{11}}\frac{\nabla_{12}}{\nabla_{22}}} \\ &= \frac{1}{\nabla_{11}}\frac{1}{1-\text{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} && (38) \\ \implies \left|\left\{\nabla^{-1}\right\}_{11}\right| &= \frac{1}{|\nabla_{11}|}\frac{1}{1-\text{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} && (39) \end{aligned}$$

The first line used the definition of ϕ; the second line used $\frac{\nabla_{21}}{\nabla_{11}}\frac{\nabla_{12}}{\nabla_{22}} = \text{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}$ from equation (34), and $\delta_2 = \delta_{-1}$; the last line applied the modulus operator, and used $\delta_1\delta_{-1} < 1$. By equation (39), we see that the matrix from equation (33) achieves the upper bound in equation (6) for $i = 1$ when $\text{sgn}(\nabla_{12}\nabla_{21}) = \text{sgn}(\nabla_{11}\nabla_{22})$, and achieves the lower bound when $\text{sgn}(\nabla_{12}\nabla_{21}) = -\text{sgn}(\nabla_{11}\nabla_{22})$.

Manipulating the expression for the $(i = 1, j \neq 1)$ elements of equation (37)

$$\begin{aligned} j \neq 1: \quad \left\{\nabla^{-1}\right\}_{1j} &= -\frac{\nabla_{1j}}{\nabla_{jj}}\frac{\nabla_{22}}{\phi} \\ \implies \left|\left\{\nabla^{-1}\right\}_{1j}\right| &= \delta_j\left|\left\{\nabla^{-1}\right\}_{11}\right| && (40) \end{aligned}$$

where the second line applied the modulus operator and used $\delta_j = \left|\frac{\nabla_{1j}}{\nabla_{jj}}\right|$ from equation (34) and $\frac{\nabla_{22}}{\phi} = \left\{\nabla^{-1}\right\}_{11}$ from equation (37). By equation (40), we see that the matrix from equation (33) achieves the bound in equation (7) for $i = 1$ and $\forall j \neq 1$.

Proof for Theorem 1. The comparative static for $i = 1$ under this ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ is

$$\begin{aligned}
\frac{\partial y_1}{\partial x} &= -\left\{\nabla^{-1}\right\}_{11} \frac{\partial f_1}{\partial x} - \left\{\nabla^{-1}\right\}_{12} \frac{\partial f_2}{\partial x} \\
&= \left|\left\{\nabla^{-1}\right\}_{11}\right| \left[-\operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{11}\right) \frac{\partial f_1}{\partial x} - \operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{12} \frac{\partial f_2}{\partial x}\right) \left|\frac{\{\nabla^{-1}\}_{12}}{\{\nabla^{-1}\}_{11}} \frac{\partial f_2}{\partial x}\right|\right] \\
&= \left|\left\{\nabla^{-1}\right\}_{11}\right| \left[-\operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{11}\right) \frac{\partial f_1}{\partial x} + \operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{11} \frac{\nabla_{12}}{\nabla_{22}} \frac{\partial f_2}{\partial x}\right) \delta_2 \left|\frac{\partial f_2}{\partial x}\right|\right] \\
&= \left|\left\{\nabla^{-1}\right\}_{11}\right| \left[-\operatorname{sgn}\left(\nabla_{11}\right) \frac{\partial f_1}{\partial x} + \operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}} \frac{\partial f_2}{\partial x}\right) \delta_{-1} \left|\frac{\partial f_{-1}}{\partial x}\right|\right] \\
&= \frac{1}{|\nabla_{11}|} \frac{-\operatorname{sgn}\left(\nabla_{11}\right) \frac{\partial f_1}{\partial x} + \operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}} \frac{\partial f_2}{\partial x}\right) \left|\frac{\partial f_{-1}}{\partial x}\right| \delta_{-1}}{1 - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}} \frac{\nabla_{21}}{\nabla_{11}}\right) \delta_1 \delta_{-1}}
\end{aligned} \tag{41}$$

The first line used equation (35) in equation (3); the second line factored out $|\{\nabla^{-1}\}_{11}|$ and wrote in terms of the sign operator; the third line used $k \neq 1 : \operatorname{sgn}\left(\{\nabla^{-1}\}_{1k}\right) = \operatorname{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}} \frac{\nabla_{22}}{\phi}\right) = \operatorname{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}} \left\{\nabla^{-1}\right\}_{11}\right)$ from equation (37), and $\left|\frac{\left\{\nabla^{-1}\right\}_{1k}}{\left\{\nabla^{-1}\right\}_{11}}\right| = \delta_k$ from equation (40); the fourth line used that $\delta_2 = \delta_{-1}$, $\left|\frac{\partial f_2}{\partial x}\right| = \left|\frac{\partial f_{-1}}{\partial x}\right|$ from equation (36), and $\operatorname{sgn}\left(\{\nabla^{-1}\}_{11}\right) = \operatorname{sgn}\left(\nabla_{11}\right)$ from equation (8); the fifth line used equation (39).

Suppose equation (10) holds. Then, the modulus of equation (41) is given by

$$\left|\frac{\partial y_1}{\partial x}\right| = \frac{1}{|\nabla_{11}|} \frac{\left|\frac{\partial f_1}{\partial x}\right| - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}} \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial x}\right) \left|\frac{\partial f_{-1}}{\partial x}\right| \delta_{-1}}{1 - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}} \frac{\nabla_{21}}{\nabla_{11}}\right) \delta_1 \delta_{-1}} \tag{42}$$

By equation (42), we see that the Jacobian and direct effects of equations (33) and (35) achieve the upper bound of equation (11) for $i = 1$ when $\operatorname{sgn}\left(\nabla_{12} \frac{\partial f_2}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{22} \frac{\partial f_1}{\partial x}\right)$ and $\operatorname{sgn}\left(\nabla_{21}\right) = \operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}}\right)$, and the lower bound for $i = 1$ when $\operatorname{sgn}\left(\nabla_{12} \frac{\partial f_2}{\partial x}\right) = \operatorname{sgn}\left(\nabla_{22} \frac{\partial f_1}{\partial x}\right)$ and $\operatorname{sgn}\left(\nabla_{21}\right) = -\operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}}\right)$.

Suppose equation (10) does not hold. Then, by equation (41), the Jacobian and direct effects of equations (33) and (35) achieve the upper bound of equation (13) for $i = 1$ when $\operatorname{sgn}\left(\nabla_{12} \frac{\partial f_2}{\partial x}\right) = \operatorname{sgn}\left(\nabla_{11} \nabla_{22}\right)$ and $\operatorname{sgn}\left(\nabla_{21}\right) = \operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}}\right)$, and lower bound for $i = 1$ when $\operatorname{sgn}\left(\nabla_{12} \frac{\partial f_2}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{11} \nabla_{22}\right)$ and $\operatorname{sgn}\left(\nabla_{21}\right) = \operatorname{sgn}\left(\nabla_{11} \frac{\nabla_{12}}{\nabla_{22}}\right)$. $\square$

A.2 Proof of theorem 1

Proof. Start with rearranging equation (3) and taking absolute values,

$$\begin{aligned}
\left|\frac{\partial y_i}{\partial x} + \left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x}\right| &= \left|\sum_{j\in\mathcal{N}\setminus i} \left\{\nabla^{-1}\right\}_{ij} \frac{\partial f_j}{\partial x}\right| \\
&\leq \sum_{j\in\mathcal{N}\setminus i} \left|\left\{\nabla^{-1}\right\}_{ij}\right| \left|\frac{\partial f_j}{\partial x}\right| \\
&\leq \left|\left\{\nabla^{-1}\right\}_{ii}\right| \sum_{j\in\mathcal{N}\setminus i} \delta_j \left|\frac{\partial f_j}{\partial x}\right| \\
&\leq \left|\left\{\nabla^{-1}\right\}_{ii}\right| \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|
\end{aligned} \tag{43}$$

The second line used the triangle inequality; the third line applied equation (7). The last line used $\forall j \neq i : \delta_j \leq \delta_{-i}$ and equation (9). The last line is equivalent to

$$\begin{aligned}
\frac{\partial y_i}{\partial x} &\in \left[- \left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x} - \left|\left\{\nabla^{-1}\right\}_{ii}\right| \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|, - \left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x} + \left|\left\{\nabla^{-1}\right\}_{ii}\right| \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|\right] \\
&\in \left|\left\{\nabla^{-1}\right\}_{ii}\right| \left[-\mathrm{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right) \frac{\partial f_i}{\partial x} - \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|, -\mathrm{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right) \frac{\partial f_i}{\partial x} + \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|\right]
\end{aligned} \tag{44}$$

The first line used the identity: $|a+b| < |c|$ is equivalent to $a \in [-b - |c|, -b + |c|]$ for any real a, b, c. The second line factored out $|\{\nabla^{-1}\}_{ii}|$.

Suppose equation (10) holds. The sign of both upper and lower bounds is determined by $-\mathrm{sgn}\left(\{\nabla^{-1}\}_{ii} \frac{\partial f_i}{\partial x}\right)$, which is equal to $-\mathrm{sgn}\left(\nabla_{ii} \frac{\partial f_i}{\partial x}\right)$ by equation (8). Hence, proving equation (12). Both bounds having equal sign implies the magnitude satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \left|\left\{\nabla^{-1}\right\}_{ii}\right| \left[\left|\frac{\partial f_i}{\partial x}\right| - \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|, \left|\frac{\partial f_i}{\partial x}\right| + \delta_{-i} \left|\frac{\partial f_{-i}}{\partial x}\right|\right] \tag{45}$$

Applying the lower (upper) bound of $|\{\nabla^{-1}\}_{ii}|$ from equation (6) to the lower (upper) bound of equation (45) proves equation (11).

Suppose equation (10) does not hold. The lower (upper) bound of equation (44) is always negative (positive), hence the comparative static sign is not determined. Thus, for both the lower and upper bound in equation (44), we must use the upper bound on $|\{\nabla^{-1}\}_{ii}|$ from equation (6), which proves equation (13). □

B Variants Details (Online Appendix)

B.1 Conditioning on More Information

The following proposition presents bounds conditional on knowledge of all the direct effects, $\{\frac{\partial f_k}{\partial x}\}_{k\in\mathcal{N}}$, as opposed to only $\frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$.

Proposition 1. (Comparative Static Bounds using all Direct Effects). *Suppose ∇ satisfies assumption 1. If for $i \in \mathcal{N}$*

$$\left|\frac{\partial f_i}{\partial x}\right| > \sum_{k\in\mathcal{N}\setminus i} \delta_k \left|\frac{\partial f_k}{\partial x}\right| \tag{46}$$

then, the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_i}\right] \tag{47}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{48}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left[\frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_{-i}}, \frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right] \tag{49}$$

with both ∇ and $\frac{\partial f}{\partial x}$ evaluated at $\boldsymbol{y} = \boldsymbol{y}^*(x)$. Conditional on $\left\{\nabla_{jj}, \delta_j, \frac{\partial f_j}{\partial x}\right\}_{j\in\mathcal{N}}$, the bounds in equations (47) and (49) are sharp.

Proof. Beginning from equation (43), and applying the identity: $|a+b| < |c|$ is equivalent to $a \in [-b-|c|, -b+|c|]$ for any real a, b, c,

$$\begin{aligned}\frac{\partial y_i}{\partial x} &\in \left[-\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} - \left|\left\{\nabla^{-1}\right\}_{ii}\right|\sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|, -\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} + \left|\left\{\nabla^{-1}\right\}_{ii}\right|\sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|\right]\\ &\in \left|\left\{\nabla^{-1}\right\}_{ii}\right|\left[-\operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right)\frac{\partial f_i}{\partial x} - \sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|, -\operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right)\frac{\partial f_i}{\partial x} + \sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|\right]\end{aligned} \tag{50}$$

The second line factored out $|\{\nabla^{-1}\}_{ii}|$. Suppose equation (46) holds, then the sign of both upper and lower bounds is determined by $-\text{sgn}\left(\{\nabla^{-1}\}_{ii}\frac{\partial f_i}{\partial x}\right)$, which is equal to $-\text{sgn}\left(\nabla_{ii}\frac{\partial f_i}{\partial x}\right)$ by equation (8). Hence, proving equation (12). Both bounds having equal sign implies the magnitude satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \left|\{\nabla^{-1}\}_{ii}\right| \left[\left|\frac{\partial f_i}{\partial x}\right| - \sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|, \left|\frac{\partial f_i}{\partial x}\right| + \sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right|\right] \tag{51}$$

Applying the lower (upper) bound of $|\{\nabla^{-1}\}_{ii}|$ from equation (6) of Lemma 1 to the lower (upper) bound of equation (51) proves equation (47).

Suppose equation (46) does not hold. The lower (upper) bound of equation (50) is always negative (positive), and hence the sign of the comparative static is not determined. Thus, for both the lower and upper bound in equation (50), we must use the upper bound on $|\{\nabla^{-1}\}_{ii}|$ from equation (6), which proves equation (49).

The bounds are proved to be sharp analogously to in section A.1. Consider the Jacobian from equation (33) and choose $\delta_{-1} = \delta_2$. The $i = 1$ comparative static is

$$\begin{aligned}
\frac{\partial y_1}{\partial x} &= \left|\{\nabla^{-1}\}_{11}\right| \left[-\text{sgn}\left(\{\nabla^{-1}\}_{11}\right)\frac{\partial f_1}{\partial x} - \sum_{k\in\mathcal{N}\setminus 1}\text{sgn}\left(\{\nabla^{-1}\}_{1k}\frac{\partial f_k}{\partial x}\right)\left|\frac{\{\nabla^{-1}\}_{1k}}{\{\nabla^{-1}\}_{11}}\frac{\partial f_k}{\partial x}\right|\right] \\
&= \left|\{\nabla^{-1}\}_{11}\right| \left[-\text{sgn}\left(\{\nabla^{-1}\}_{11}\right)\frac{\partial f_1}{\partial x} + \sum_{k\in\mathcal{N}\setminus 1}\text{sgn}\left(\{\nabla^{-1}\}_{11}\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\partial f_k}{\partial x}\right)\left|\frac{\{\nabla^{-1}\}_{1k}}{\{\nabla^{-1}\}_{11}}\frac{\partial f_k}{\partial x}\right|\right] \\
&= \left|\{\nabla^{-1}\}_{11}\right| \left[-\text{sgn}\left(\nabla_{11}\right)\frac{\partial f_1}{\partial x} + \sum_{k\in\mathcal{N}\setminus 1}\text{sgn}\left(\nabla_{11}\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\partial f_k}{\partial x}\right)\delta_k\left|\frac{\partial f_k}{\partial x}\right|\right] \\
&= \frac{1}{|\nabla_{11}|}\frac{-\text{sgn}\left(\nabla_{11}\right)\frac{\partial f_1}{\partial x} + \sum_{k\in\mathcal{N}\setminus 1}\text{sgn}\left(\nabla_{11}\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\partial f_k}{\partial x}\right)\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1 - \text{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}}
\end{aligned} \tag{52}$$

The first line factored out $|\{\nabla^{-1}\}_{11}|$ from equation (3); the second line used $k \neq 1 : \text{sgn}\left(\{\nabla^{-1}\}_{1k}\right) = \text{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\nabla_{22}}{\phi}\right) = \text{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}}\{\nabla^{-1}\}_{11}\right)$ from equation (37), and $\left|\frac{\{\nabla^{-1}\}_{1k}}{\{\nabla^{-1}\}_{11}}\right| = \delta_k$ from equation (40); the third line used $\text{sgn}\left(\{\nabla^{-1}\}_{11}\right) = \text{sgn}\left(\nabla_{11}\right)$ from equation (8); the fourth line used equation (39).

Suppose equation (10) holds. Then, the absolute of equation (52) is given by

$$\left|\frac{\partial y_1}{\partial x}\right| = \frac{1}{|\nabla_{11}|}\frac{\left|\frac{\partial f_1}{\partial x}\right| + \sum_{k\in\mathcal{N}\setminus 1}\operatorname{sgn}\left(\nabla_{11}\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\partial f_k}{\partial x}\right)\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} \tag{53}$$

Conditional on $\left\{\nabla_{ii}, \delta_i, \frac{\partial f_i}{\partial x}\right\}_{i\in\mathcal{N}}$, the comparative static achieves the upper bound of equation (47) for $i = 1$ when $\forall k \neq 1 : \operatorname{sgn}(\nabla_{1k}) = \operatorname{sgn}\left(\nabla_{11}\nabla_{kk}\frac{\partial f_k}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{11}\nabla_{22}}\right)$, and the lower bound when $\forall k \neq 1 : \operatorname{sgn}(\nabla_{1k}) = -\operatorname{sgn}\left(\nabla_{11}\nabla_{kk}\frac{\partial f_k}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = -\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{11}\nabla_{22}}\right)$.

Suppose equation (10) does not hold. Then, by equation (52), conditional on $\left\{\nabla_{ii}, \delta_i, \frac{\partial f_i}{\partial x}\right\}_{i\in\mathcal{N}}$, the comparative static achieves the upper bound of equation (49) for $i = 1$, when $\forall k \neq 1 : \operatorname{sgn}(\nabla_{1k}) = \operatorname{sgn}\left(\nabla_{11}\nabla_{kk}\frac{\partial f_k}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{11}\nabla_{22}}\right)$, and the lower bound when $\forall k \neq 1 : \operatorname{sgn}(\nabla_{1k}) = -\operatorname{sgn}\left(\nabla_{11}\nabla_{kk}\frac{\partial f_k}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{11}\nabla_{22}}\right)$. □

B.2 Conditioning on Less Information

The following proposition presents bounds without conditioning on knowledge of δ_{-i}.

Proposition 2. (Comparative Static Bounds without δ_{-i}). *Suppose ∇ satisfies assumption 1. Then, if for $i \in \mathcal{N}$,*

$$\left|\frac{\partial f_i}{\partial x}\right| \geq \left|\frac{\partial f_{-i}}{\partial x}\right| \tag{54}$$

the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left(\frac{\left|\frac{\partial f_i}{\partial x}\right| - \left|\frac{\partial f_{-i}}{\partial x}\right|}{1+\delta_i}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}\right) \tag{55}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}(\nabla_{ii})\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{56}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left(\frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}, \frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}\right) \tag{57}$$

with $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$ *evaluated at* $\mathbf{y} = \mathbf{y}^*(x)$. *Conditional on* $\nabla_{ii}, \delta_i, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$, *the bounds are sharp.*

Proof. Equations (55) and (57) are implied by setting $\delta_{-i} = 1$ in equations (11) and (12), respectively. The bounds become open because only $\delta_{-i} < 1$ is permitted under assumption 1. Equation (56) is implied for the same reason equation (12) is implied in theorem 1.

The bounds are proved sharp by using the same proof as for theorem 1 in section A.1, except by choosing $\delta_2 = 1$, which we are now free to choose because only δ_1 is being conditioned on. □

B.3 Row Diagonal Dominance

The following proposition presents bounds using row diagonal dominance (equation 58) as opposed to column diagonal dominance (assumption 1). Note that the iDD degree (equation 63) is now calculated using the row sum rather than the column sum, a stronger condition on $\left|\frac{\partial f_{-i}}{\partial x}\right|$ is required (equation 59), and the effect of the shock on other nodes is characterized (62).

Proposition 3. (Comparative Static Bounds under Row Diagonal Dominance) *Suppose* ∇ *satisfies*

$$\forall i \in \mathcal{N}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ij}| \tag{58}$$

If for $i \in \mathcal{N}$

$$\left|\frac{\partial f_{-i}}{\partial x}\right| = 0 \tag{59}$$

then,

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right|}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right] \tag{60}$$

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{61}$$

and for $\forall j \in \mathcal{N}\backslash i$

$$\left|\frac{\partial y_j}{\partial x}\right| \le \delta_j \left|\frac{\partial y_i}{\partial x}\right| \tag{62}$$

where

$$\delta_i \equiv \frac{\sum_{j\in\mathcal{N}\backslash i}|\nabla_{ij}|}{|\nabla_{ii}|} \tag{63}$$

with both ∇ *and* $\frac{\partial \boldsymbol{f}}{\partial x}$ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$*. Conditional on* $\left\{\nabla_{jj}, \delta_j\right\}_{j\in\mathcal{N}}, \frac{\partial f_i}{\partial x}$*, equations (60) and (62) are sharp.*

Proof. Lemma 1 applies to ∇ satisfying row diagonal dominance, equation (58) with the only two differences: δ_i is calculated using the row sum, equation (63), and equation (7) is replaced by

$$\left|\left\{\nabla^{-1}\right\}_{ji}\right| \le \delta_j \left|\left\{\nabla^{-1}\right\}_{ii}\right| \tag{64}$$

Turning to the comparative statics. Applying equation (59) in equation (3)

$$\frac{\partial y_i}{\partial x} = -\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} \tag{65}$$

as equation (59) implies $\forall j \in \mathcal{N}\backslash i : \frac{\partial f_j}{\partial x} = 0$. The comparative static sign is

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\frac{\partial f_i}{\partial x}\right)$$

where the second equality used (8). Thus proving equation (61). The magnitude of the comparative static

$$\begin{aligned} \left|\frac{\partial y_i}{\partial x}\right| &= \left|\left\{\nabla^{-1}\right\}_{ii}\right| \left|\frac{\partial f_i}{\partial x}\right| \\ &\in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right|}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right] \end{aligned} \tag{66}$$

where the second line used equation (6). Thus proving equation (60). Consider the effect on $j \neq i$

$$\begin{aligned} \frac{\partial y_j}{\partial x} &= -\left\{\nabla^{-1}\right\}_{ji} \frac{\partial f_i}{\partial x} \\ \left|\frac{\partial y_j}{\partial x}\right| &= \left|\left\{\nabla^{-1}\right\}_{ji} \frac{\partial f_i}{\partial x}\right| \\ &\leq \delta_j \left|\left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x}\right| \\ &= \delta_j \left|\frac{\partial y_i}{\partial x}\right| \end{aligned}$$

where the third line used equation (64), and the last line used equation (66). Thus proving equation (62)

Equations (60) and (62) being sharp follow from equations (6) and (7) being sharp in lemma 1, respectively, as the only difference is a multiple of $\left|\frac{\partial f_i}{\partial x}\right|$, which is part of the information set in this proposition. □

B.4 Generalized Diagonal Dominance: Sub-Unity Spectral Radius

The following proposition presents an example of generalized diagonal dominance using the spectral radius of a transformed Jacobian, given by

$$A_{ij} \equiv \begin{cases} i = j & 0 \\ i \neq j & \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| \end{cases} \tag{67}$$

Denote by ρ the spectral radius of A, and by v_i a left-eigenvector of A with eigenvalue equal in magnitude to ρ. Set $g_i' = v_i$. If A is irreducible and $\rho < 1$, then $g_i'\nabla_{ij}$ satisfies assumption 1. Proposition 4 proves this and derives the implied comparative static bounds, in terms of ρ and v_i. Note that this result conditions on more information, specifically all of the direct effects, as in proposition 1.

Proposition 4. (Comparative Static Bounds under Sub-Unity Spectral Radius). *Suppose A is irreducible and $\rho < 1$. If for $i \in \mathcal{N}$*

$$\left| \frac{\partial f_i}{\partial x} \right| > \rho \sum_{k \neq i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right| \tag{68}$$

then, the magnitude of the comparative static satisfies

$$\left| \frac{\partial y_i}{\partial x} \right| \in \frac{1}{|\nabla_{ii}|} \left[\frac{\left| \frac{\partial f_i}{\partial x} \right| - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1+\rho^2}, \frac{\left| \frac{\partial f_i}{\partial x} \right| + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1-\rho^2} \right] \tag{69}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x} \right) = -\operatorname{sgn}\left(\nabla_{ii} \right) \operatorname{sgn}\left(\frac{\partial f_i}{\partial x} \right) \tag{70}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|} \left[\frac{-\operatorname{sgn}\left(\nabla_{ii} \right) \frac{\partial f_i}{\partial x} - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1-\rho^2}, \frac{-\operatorname{sgn}\left(\nabla_{ii} \right) \frac{\partial f_i}{\partial x} + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1-\rho^2} \right] \tag{71}$$

Where $\rho > 0$, and $\{v_i\}_{i \in \mathcal{N}}$ are unique and strictly positive, and both ∇ and $\frac{\partial f}{\partial x}$ are evaluated at $\boldsymbol{y} = \boldsymbol{y}^* (x)$.

Proof. Irreducibility of A and $\rho < 1$ is not sufficient for ∇_{ij} to satisfy assumption 1, thus we cannot directly apply theorem 1 (or proposition 1). However, it is sufficient for $\tilde{\nabla}_{ij} \equiv v_i \nabla_{ij}$ to satisfy assumption 1 (implying ∇ is generalized diagonally dominant, see remark 4). Thus, we can rewrite the comparative statics in terms of $\tilde{\nabla}$ and apply proposition 1.

To prove that $\tilde{\nabla}$ satisfies assumption 1, I rely on the Perron-Frobenius theorem (Horn and Johnson, 2012 theorem 8.4.4), which applies to A because A is nonnegative and irreducible. The theorem implies that ρ is strictly positive, that it is an eigenvalue of A, and its associated left-eigenvector $\{v_i\}_{i \in \mathcal{N}}$ is unique and has all elements strictly positive, $\forall i : v_i > 0$. Utilizing this,

$$\begin{aligned}
\left|\tilde{\nabla}_{ii}\right| - \sum_{j\neq i}\left|\tilde{\nabla}_{ji}\right| &= \left|v_i\nabla_{ii}\right| - \sum_{j\neq i}\left|v_j\nabla_{ji}\right| \\
&= v_i\left|\nabla_{ii}\right| - \sum_{j\neq i} v_j\left|\nabla_{ji}\right| \\
&= \left|\nabla_{ii}\right|\left(v_i - \sum_{j\neq i} v_j\frac{\left|\nabla_{ji}\right|}{\left|\nabla_{ii}\right|}\right) \\
&= \left|\nabla_{ii}\right|\left(v_i - \sum_{j} v_j A_{ji}\right) \\
&= \left|\nabla_{ii}\right| v_i\left(1-\rho\right) \\
&> 0
\end{aligned}$$

The first line used the definition of $\tilde{\nabla}_{ij} \equiv v_i\nabla_{ij}$; the second line used that $\forall i : v_i > 0$; the fourth line used the definition of A, equation (67); the fifth line used that ρ is eigenvalue of A with eigenvector $\{v_i\}_{i\in\mathcal{N}}$; and the last line used that $\rho < 1$. Thus, $\tilde{\nabla}$ is strictly column diagonally dominant and therefore satisfies assumption 1. The iDD degree of $\tilde{\nabla}$, which I denote by $\tilde{\delta}$, is

$$\begin{aligned}
\tilde{\delta}_i &\equiv \frac{\sum_{j\neq i}\left|\tilde{\nabla}_{ji}\right|}{\left|\tilde{\nabla}_{ii}\right|} \\
&= \frac{\sum_{j\neq i}\left|v_j\nabla_{ji}\right|}{\left|v_i\nabla_{ii}\right|} \\
&= \frac{\sum_{j\neq i} v_j\frac{\left|\nabla_{ji}\right|}{\left|\nabla_{ii}\right|}}{v_i} \\
&= \frac{\sum_{j} v_j A_{ji}}{v_i} \\
&= \frac{v_i\rho}{v_i} \\
&= \rho
\end{aligned}$$

The first line used the definition of the iDD degree, definition 1; the second line used the definition of $\tilde{\nabla}_{ij} \equiv v_i\nabla_{ij}$; the third line used that $\forall i : v_i > 0$; the forth line the definition of A, equation (67); the fifth line used that ρ is eigenvalue of A with eigenvector $\{v_i\}_{i\in\mathcal{N}}$. Next, I rewrite the comparative statics in terms of $\tilde{\nabla}$

$$\begin{aligned}\frac{\partial y_i}{\partial x} &= -\sum_{k\in\mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial x} \\ &= -\sum_{k\in\mathcal{N}} \left\{\tilde{\nabla}^{-1}\right\}_{ik} \frac{\partial \tilde{f}_k}{\partial x}\end{aligned}$$

where I used $\frac{\partial \tilde{f}_k}{\partial x} \equiv v_k \frac{\partial f_k}{\partial x}$. Now, we apply proposition 1 with $\tilde{\nabla}_{ij}$ and $\frac{\partial \tilde{f}_k}{\partial x}$ in place of ∇_{ij} and $\frac{\partial f_k}{\partial x}$. First, equation (46) becomes

$$\begin{aligned} & \left|\frac{\partial \tilde{f}_i}{\partial x}\right| > \sum_{k\in\mathcal{N}\setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right| \\ \iff \quad & \left|\frac{\partial f_i}{\partial x}\right| > \rho \sum_{k\neq i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right| \end{aligned} \tag{72}$$

Equation (47) becomes

$$\begin{aligned}\left|\frac{\partial y_i}{\partial x}\right| &\in \frac{1}{\left|\tilde{\nabla}_{ii}\right|} \left[\frac{\left|\frac{\partial \tilde{f}_i}{\partial x}\right| - \sum_{k\in\mathcal{N}\setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1+\tilde{\delta}_i\tilde{\delta}_{-i}}, \frac{\left|\frac{\partial \tilde{f}_i}{\partial x}\right| + \sum_{k\in\mathcal{N}\setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1-\tilde{\delta}_i\tilde{\delta}_{-i}}\right] \\ &= \frac{1}{\left|\nabla_{ii}\right|} \left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \rho\sum_{k\in\mathcal{N}\setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1+\rho^2}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \rho\sum_{k\in\mathcal{N}\setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1-\rho^2}\right]\end{aligned}$$

where in the second line I used $\tilde{\nabla}_{ij} \equiv v_i \nabla_{ij}$ and $\frac{\partial \tilde{f}_k}{\partial x} \equiv v_k \frac{\partial f_k}{\partial x}$. The sign of the comparative static, equation (48) remains the same, because

$$\begin{aligned}\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) &= -\operatorname{sgn}\left(\tilde{\nabla}_{ii}\right)\operatorname{sgn}\left(\frac{\partial \tilde{f}_i}{\partial x}\right) \\ &= -\operatorname{sgn}\left(v_i\nabla_{ii}\right)\operatorname{sgn}\left(v_i\frac{\partial f_i}{\partial x}\right) \\ &= -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right)\end{aligned}$$

given $\forall i : v_i > 0$. Equation (49) becomes

$$\frac{\partial y_i}{\partial x} \in \frac{1}{\left|\tilde{\nabla}_{ii}\right|}\left[\frac{-\text{sgn}\left(\tilde{\nabla}_{ii}\right)\frac{\partial \tilde{f}_i}{\partial x} - \sum_{k\in\mathcal{N}\setminus i}\tilde{\delta}_k\left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1-\tilde{\delta}_i\tilde{\delta}_{-i}}, \frac{-\text{sgn}\left(\tilde{\nabla}_{ii}\right)\frac{\partial \tilde{f}_i}{\partial x} + \sum_{k\in\mathcal{N}\setminus i}\tilde{\delta}_k\left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1-\tilde{\delta}_i\tilde{\delta}_{-i}}\right]$$

$$= \frac{1}{\left|\nabla_{ii}\right|}\left[\frac{-\text{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \rho\sum_{k\in\mathcal{N}\setminus i}\frac{v_k}{v_i}\left|\frac{\partial f_k}{\partial x}\right|}{1-\rho^2}, \frac{-\text{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \rho\sum_{k\in\mathcal{N}\setminus i}\frac{v_k}{v_i}\left|\frac{\partial f_k}{\partial x}\right|}{1-\rho^2}\right]$$

□

B.5 Signed Diagonal Dominance

The following proposition presents bounds that also condition on the sign on the elements of $\nabla, \frac{\partial f}{\partial x}$. For ∇, all diagonal elements have the same sign, and each of the off-diagonal elements have a sign that is not equal to that of the diagonal elements (so either the opposite sign, or zero). For $\frac{\partial f_j}{\partial x}$, all $j \neq i$ have the same sign, which is different to $j = i$. These are equations (73) and (74) in the following proposition. Note that proposition 5 only applies in the case where equation (46) holds.

Proposition 5. (Comparative Static Bounds under Signed Diagonal Dominance). *Suppose assumption 1 holds, and equation (46) holds for $i \in \mathcal{N}$. Also suppose*

$$\forall k, j \neq k: \quad s_1 = \text{sgn}\left(\nabla_{kk}\right) \neq \text{sgn}\left(\nabla_{kj}\right) \tag{73}$$

$$\forall j \neq i: \quad s_2 = \text{sgn}\left(\frac{\partial f_i}{\partial x}\right) \neq \text{sgn}\left(\frac{\partial f_j}{\partial x}\right) \tag{74}$$

holds for some $s_1 \in \{-1, 1\}, s_2 \in \{-1, 1\}$. Then,

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{\left|\nabla_{ii}\right|}\left[\left|\frac{\partial f_i}{\partial x}\right| - \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|, \frac{\left|\frac{\partial f_i}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right] \tag{75}$$

and

$$\text{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{76}$$

with both ∇ and $\frac{\partial f}{\partial x}$ evaluated at $\boldsymbol{y} = \boldsymbol{y}^*(x)$.

Proof. Equation (73), combined with ∇ being diagonally dominant (assumption 1), implies $s_1\nabla$ is an M-matrix (Horn and Johnson, 1991, chapter 2.5), which has the following two properties

$$\forall i: \quad s_1 \left\{\nabla^{-1}\right\}_{ii} \geq \frac{1}{s_1 \nabla_{ii}} \tag{77}$$

$$\forall i, j \neq i: \quad s_1 \left\{\nabla^{-1}\right\}_{ij} \geq 0 \tag{78}$$

To prove these, rewrite the Jacobian as

$$\nabla_{ij} \equiv \left(I_{ij} - A_{ij}\right) \nabla_{jj} \tag{79}$$

where $A_{ij} \equiv \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| (1 - I_{ij}) \geq 0$. Equation (79) follows by noting that $\forall i, j \neq i$: $\frac{\nabla_{ij}}{\nabla_{jj}} \leq 0$ due to equation (73). Diagonal dominance of ∇ implies the spectral radius of A, denoted ρ, is less than 1: $\rho \leq \max_j \sum_i |A_{ij}| = \max_j \sum_{i \neq j} \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| < 1$, where the first inequality follows using the Gerschgorin Circle theorem, and the last inequality follows from diagonal dominance of ∇. Thus, one can apply the Neumann expansion $(I - A)^{-1} = I + A + A^2 + \cdots$, which is valid under $\rho < 1$ (Johnson, 1982), implying

$$\forall i: \quad \left\{(I - A)^{-1}\right\}_{ii} \geq 1$$
$$\forall i, j \neq i: \quad \left\{(I - A)^{-1}\right\}_{ji} \geq 0$$

which follows by noting that $\forall i, j : A_{ij} \geq 0$. Using these facts in the matrix inverse of equation (79) implies equations (77) because

$$\forall i: \quad s_1 \left\{\nabla^{-1}\right\}_{ii} = \frac{1}{s_1 \nabla_{ii}} \left\{(I - A)^{-1}\right\}_{ii} \geq \frac{1}{s_1 \nabla_{ii}}$$

and (78) because

$$\forall i, j \neq i: \quad s_1 \left\{\nabla^{-1}\right\}_{ij} = \frac{1}{s_1 \nabla_{ii}} \left\{(I - A)^{-1}\right\}_{ij} \geq 0$$

noting that $s_1 \nabla_{ii} > 0$. Turning to the comparative static, rewrite equation (3) as

$$\frac{\partial y_i}{\partial x} = - \left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x} \underbrace{\left(1 + \sum_{k \in \mathcal{N} \setminus i} \frac{\left\{\nabla^{-1}\right\}_{ik}}{\left\{\nabla^{-1}\right\}_{ii}} \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}}\right)}_{(*)} \tag{80}$$

The term $(*)$ can be bounded as follows

$$1 + \sum_{k \in \mathcal{N} \backslash i} \underbrace{\frac{\{\nabla^{-1}\}_{ik}}{\{\nabla^{-1}\}_{ii}}}_{\geq 0} \underbrace{\frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}}}_{\leq 0} \in \left[\underbrace{1 - \sum_{k \in \mathcal{N} \backslash i} \delta_k \left| \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}} \right|}_{\geq 0}, 1 \right] \tag{81}$$

where $\forall k \in \mathcal{N} \backslash i : \frac{\{\nabla^{-1}\}_{ik}}{\{\nabla^{-1}\}_{ii}} \geq 0$ from equations (77), (78); $\forall k \in \mathcal{N} \backslash i : \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}} \leq 0$ from equation (74). The lower bound used equation (7), and is positive by equation (46).

Taking the absolute value of equation (80) and using equation (81)

$$\left| \frac{\partial y_i}{\partial x_i} \right| \in \left| \{\nabla^{-1}\}_{ii} \right| \left| \frac{\partial f_i}{\partial x} \right| \left[1 - \sum_{k \in \mathcal{N} \backslash i} \delta_k \left| \frac{\partial f_k}{\partial x} / \frac{\partial f_i}{\partial x} \right|, 1 \right] \tag{82}$$

$|\{\nabla^{-1}\}_{ii}|$ can be bounded below using equation (77) and above using equation (6)

$$\forall i: \quad \left| \{\nabla^{-1}\}_{ii} \right| \in \left[\frac{1}{|\nabla_{ii}|}, \frac{1}{|\nabla_{ii}|} \frac{1}{1 - \delta_i \delta_{-i}} \right] \tag{83}$$

Inserting the lower (upper) bound into the lower (upper) bound of equation (82) proves equation (75). Equation (76) holds by proposition 1 as both assumption 1 and equation (46) hold. □

C Model Details (Online Appendix)

Detail of the models listed in table 1.

C.1 Production Network

Consider the workhorse production network from Carvalho and Tahbaz-Salehi (2019), section 2.1. This model is not only highly tractable and thus useful for expositional purposes, but it also highly relevant, underlying many macroeconomic frameworks in the recent literature, as attention has begun to focus on the macroeconomic implications of production networks.[29]

Nodes in this framework correspond to firms, and interactions between nodes are described the by the supply chain network. For the comparative static on firm production, I show that assumption 1, and the additional conditions required for proposition 5, are always satisfied. The bounds for the comparative statics with respect to a government expenditure shock, a firm-level fiscal multiplier, depend only the expenditure share on intermediate inputs by firms, and the share of expenditure on each firm by households. Notably, no supply chain data is required for the bounds, which is potentially quite powerful, because such data in simply unavailable in most settings (Pichler et al., 2023). The bounds reveal a highly succinct necessary condition for the fiscal multiplier on a firm to be greater than one: the expenditure share on intermediate goods by the firm is greater than the share of household expenditure on that firm.

There are $i \in \mathcal{N}$ firms, each with production function

$$q_i = \zeta_i l_i^{1-\alpha_i} \prod_{j \in \mathcal{N}} x_{ji}^{A_{ji}}$$

where q_i is output, l_i is labor, and x_{ji} are intermediate inputs purchased from firm j. $\{A_{ji}\}_{i \in \mathcal{N}, j \in \mathcal{N}}$ is the input-output matrix: the share of firm i total expenditure spent on goods from firm j. Following the typical convention in the literature, I assume $\forall i \in \mathcal{N} : A_{ii} = 0$ (the firm doesn't buy from itself). $\sum_{j \in \mathcal{N} \setminus i} A_{ji} \equiv \alpha_i \in (0, 1)$ is

[29] For example. see Acemoglu et al. (2012, 2016); Baqaee (2018) for the macro amplification of micro shocks; Afrouzi and Bhattarai (2023); Ghassibe (2021); Rubbo (2023) for applications in monetary economics; Flynn et al. (2023) for fiscal multipliers; and Baqaee and Farhi (2021); Bonadio et al. (2021) for applications to covid 2019. For early applications see Long and Plosser (1983).

the expenditure share on intermediate inputs, with $1-\alpha_i$ the labor share. $\zeta_i \equiv (1-\alpha_i)^{-(1-\alpha_i)} \prod_{j\in\mathcal{N}} (A_{ji})^{-A_{ji}}$ is a normalization constant. The firms takes prices $\{p_j\}_{j\in\mathcal{N}}$ and the wage $w \equiv 1$ (the numeraire) as given, due to perfect competition.

There is a representative household who supplies labor to all the firms, consumes their goods according to a Cobb-Douglas utility function, and pays taxes T to the government. Demand from the household for good i is therefore $c_i = \frac{1}{p_i}\beta_i (w-T)$, where $\beta_i \in [0,1]$ is the share of household expenditure on good i, and $\sum_{i\in\mathcal{N}} \beta_i = 1$.

The government exogenously demands g_i of the goods from firm i. The total demand facing firm i is therefore

$$q_i = c_i + g_i + \sum_{j\in\mathcal{N}} x_{ij} \tag{84}$$

The government balances its budget, $T = \sum_{i\in\mathcal{N}} p_i g_i$. Consider the comparative static of the endogenous firm output q_i with respect to the exogenous government expenditure g_j — a firm-level fiscal multiplier. Then, we can use the equilibrium conditions to derive the following equations of state for the endogenous states $\{q_i\}_{i\in\mathcal{N}}$, given exogenous shocks $\{g_i\}_{i\in\mathcal{N}}$[30]

$$0 = f_i(\mathbf{q},\mathbf{g}) = q_i - \beta_i \left(1 - \sum_{j\in\mathcal{N}} g_j\right) - g_i - \sum_{j\in\mathcal{N}} A_{ij} q_j \tag{85}$$

The Jacobian and direct effects of this model are

$$\nabla_{ik} = I_{ik} - A_{ik}, \qquad \frac{\partial f_k}{\partial g_j} = -I_{kj} + \beta_k \tag{86}$$

The comparative static, equation (3), is

$$\frac{\partial q_i}{\partial g_j} = -\sum_{k\in\mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial g_j} = \sum_{k\in\mathcal{N}} \left\{(I-A)^{-1}\right\}_{ik} (I_{kj} - \beta_k) \tag{87}$$

where $(I-A)^{-1}$ is the Leontief Inverse (see Carvalho and Tahbaz-Salehi, 2019 equa-

[30]The firms' first order conditions imply the equilibrium price is $p_i = w \equiv 1$ and the equilibrium intermediate good expenditure share is $x_{ji} = A_{ji} q_i$. Inserting these, along with household demand, and the government budget balance, into equation (84), yields equation (85).

tion 5). The Jacobian ∇ is always diagonally dominant (assumption 1) in this model

$$\sum_{j\in\mathcal{N}\setminus i}|\nabla_{ji}| = \sum_{j\in\mathcal{N}\setminus i}A_{ji} = \alpha_i < 1 = |\nabla_{ii}|$$

Thus, theorem 1 applies to this model. The iDD degree (definition 1) is

$$\delta_i \equiv \frac{\sum_{j\in\mathcal{N}\setminus i}|\nabla_{ji}|}{|\nabla_{ii}|} = \alpha_i, \qquad \delta_{-i} = \max_{j\in\mathcal{N}\setminus i}\alpha_j \equiv \alpha_{-i}$$

and equal to the expenditure share on intermediate inputs, α_i. The sum of the direct effects on all other nodes is

$$\left|\frac{\partial f_{-i}}{\partial g_j}\right| \equiv \sum_{k\neq i}\left|\frac{\partial f_k}{\partial g_j}\right| = \begin{cases} j=i & \sum_{k\neq i}\beta_k = 1-\beta_i \\ j\neq i & 1-\beta_j+\sum_{k\notin\{i,j\}}\beta_k = 2\left(1-\beta_j\right)-\beta_i \end{cases}$$

The direct effects $\frac{\partial f_i}{\partial g_j}$ satisfy equation (10) for $j=i$, as

$$\delta_{-i}\left|\frac{\partial f_{-i}}{\partial g_i}\right| = \alpha_{-i}\left(1-\beta_i\right) < 1-\beta_i = \left|\frac{\partial f_i}{\partial g_i}\right|$$

Thus, for $j=i$, $\frac{\partial q_i}{\partial g_i}$, bounds on the magnitude are given by equation (11)

$$\frac{\partial q_i}{\partial g_i} \in \left[\frac{\left(1-\beta_i\right)\left(1-\alpha_{-i}\right)}{1+\alpha_i\alpha_{-i}}, \frac{\left(1-\beta_i\right)\left(1+\alpha_{-i}\right)}{1-\alpha_i\alpha_{-i}}\right] \tag{88}$$

and the sign is given by equation (12)

$$\operatorname{sgn}\left(\frac{\partial q_i}{\partial g_i}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial g_i}\right) = -\cdot 1\cdot -1 = 1$$

For $j\neq i$, $\frac{\partial q_i}{\partial g_j}$ the bounds are given by equation (13)

$$\frac{\partial q_i}{\partial g_j} \in \left[\frac{\beta_i-\left(2-2\beta_j-\beta_i\right)\alpha_{-i}}{1-\alpha_i\alpha_{-i}}, \frac{\beta_i+\left(2-2\beta_j-\beta_i\right)\alpha_{-i}}{1-\alpha_i\alpha_{-i}}\right] \tag{89}$$

Consider the bounds for $j=i$ in equation (88). Theorem 1 implies these bounds are sharp conditional on $\{\nabla_{jj},\delta_j,\}_{j\in\mathcal{N}}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$, which corresponds to $\{\alpha_j\}_{j\in\mathcal{N}}, \beta_i$ in this model. That is, if one only knew the values of $\{\alpha_j\}_{j\in\mathcal{N}}, \beta_i$ and nothing else, specifically no knowledge of the input-output matrix $\{A_{ij}\}_{i\in\mathcal{N},j\in\mathcal{N}\setminus i}$ beyond $\{\alpha_i\}_{i\in\mathcal{N}}$ or $\{\beta_j\}_{j\in\mathcal{N}\setminus i}$, equation (88) is the most one can say about the maximum and minimum values of the comparative static, $\frac{\partial q_i}{\partial g_i}$.

If one conditions on more information, one can say more about the comparative static. For instance, applying proposition 1 (see remark 1)

$$\begin{aligned}\left|\frac{\partial q_i}{\partial g_i}\right| &\in \left[\frac{1-\beta_i-\sum_{k\neq i}\alpha_k\beta_k}{1-\alpha_i\alpha_{-i}}, \frac{1-\beta_i+\sum_{k\neq i}\alpha_k\beta_k}{1-\alpha_i\alpha_{-i}}\right] \\ \frac{\partial q_j}{\partial g_i} &\in \left[\frac{-\beta_j-\sum_{k\notin\{i,j\}}\alpha_k\beta_k-\alpha_i\left(1-\beta_i\right)}{1-\alpha_i\alpha_{-i}}, \frac{-\beta_j+\sum_{k\notin\{i,j\}}\alpha_k\beta_k+\alpha_i\left(1-\beta_i\right)}{1-\alpha_i\alpha_{-i}}\right]\end{aligned} \tag{90}$$

Consider again the bounds for $j=i$, this time using equation (90). Proposition 1 implies these bounds are sharp conditional on $\left\{\nabla_{jj}, \delta_j, \frac{\partial f_j}{\partial x}\right\}_{j\in\mathcal{N}}$, which corresponds to $\{\alpha_j, \beta_j\}_{j\in\mathcal{N}}$ in this model. That is, more information is conditioned on than for the bounds in equation (88), and the result is that the bounds in equation (90) are narrower (note that $\sum_{k\neq i}\alpha_k\beta_k \leq \left(1-\beta_i\right)\alpha_{-i}$)

Alternatively, one can apply proposition 5, by using the fact that trade flows are nonnegative, $\forall i, j\neq i : A_{ij}\geq 0$. This implies that the off-diagonal elements of the Jacobian are all nonpositive, $\forall i, j\neq i : \nabla_{ij} = -A_{ij}\leq 0$, while the diagonal elements are positive $\forall i : \nabla_{ii} = 1-A_{ii} = 1 > 0$.[31] Analogously, note that the direct effect matrix $\frac{\partial f_i}{\partial g_j}$ satisfies the opposite sign pattern, $\forall i, j\neq i : \frac{\partial f_i}{\partial g_i} = -1+\beta_i < 0, \frac{\partial f_i}{\partial g_j} = \beta_i \geq 0$. Thus, ∇_{ij} and $\frac{\partial f_i}{\partial g_j}$ satisfy the additional conditions required for proposition 5, specifically equations (73) and (74). The bounds for $j=i$ in this case are

$$\frac{\partial q_i}{\partial g_i} \in \left[\left(1-\beta_i\right)\left(1-\alpha_{-i}\right), \frac{1-\beta_i}{1-\alpha_i\alpha_{-i}}\right] \tag{91}$$

Comparing equation (91) to (88), it is immediate that the bound range in (91) is narrower, again due to the proposition conditioning on a greater set of information.

The power of the bounds, from either theorem 1 or propositions 1 or 5, can be understood by comparing them to the exact comparative static, equation (87). First, they are identified with less information. As already highlighted, the bounds

[31] The off-diagonal elements of ∇ being all negative, while the diagonal elements all positive, translates into the feedback in this model being entirely positive: an increase in q_j causes q_i to increase, for any i, j. In terms of the economics, when firm j increases their output $q_j\uparrow$, they buy more goods from firm i according to the input share A_{ij}, and thus firm i output increases, $q_i\uparrow$. This is the only source of feedback in this model.

require no knowledge of the input-output matrix A beyond $\{\alpha_i\}_{i\in\mathcal{N}}$, whereas the exact comparative static requires full knowledge of A. Second, they are much simpler analytically. In particular, the bounds do not require a matrix inversion of $I - A$, unlike in the exact comparative static.

Not requiring knowledge of the full input-output matrix A is potentially quite powerful, because such data in simply unavailable in most settings (Pichler et al., 2023). The results of this paper provide a way to learn about the comparative statics in production networks without needing to observe the full supply chain network. In particular, one only needs knowledge of the low dimensional sufficient statistics, corresponding in this model to the total expenditure share on intermediate inputs by each firm, α_i and the expenditure share by households on each firm, β_i. These are firm level variables, rather than firm-firm level (e.g. input-output data), and are therefore more readily available.

Not having a matrix inversion in the expression significantly increases tractability. This allows one to derive new theoretical insights about the propagation of shocks in production networks, which the literature has gone some way already in characterizing (see e.g. Acemoglu et al., 2016; Carvalho et al., 2021). For example, a novel necessary condition for the firm-level fiscal multiplier to be greater than one, $\frac{\partial q_i}{\partial g_i} > 1$, can be simply derived from using the upper bound in equation (91)[32]

$$\alpha_i > \beta_i \tag{92}$$

This result implies that in a production network with any input-output matrix, $\{A_{ij}\}_{i\in\mathcal{N},j\in\mathcal{N}\setminus i}$, the fiscal multiplier on firm i, $\frac{\partial q_i}{\partial g_i}$, can be greater than one only if the firm's expenditure share on intermediate inputs, α_i, is greater than the expenditure share by households on the firm, β_i. Discerning this property from the comparative static exact expression, equation (87), is less straightforward, if possible at all.

C.2 New Keynesian

A classic question in macroeconomics is what is the size of the fiscal multiplier (Ramey, 2019): the change in GDP given an increase in government spending. Particular attention has been given to when is this greater than one, as this corresponds

[32]Proof: $\frac{\partial q_i}{\partial g_i} > 1 \implies \frac{1-\beta_i}{1-\alpha_i\alpha_{-i}} > 1 \implies \alpha_i\alpha_{-i} > \beta_i \implies \alpha_i > \beta_i$. The last implication used $\alpha_{-i} < 1$, in the spirit of remark 2.

to the case where private expenditure is not crowded out by the public expenditure (Christiano et al., 2011). Utilizing the recently influential intertemporal Keynesian cross framework of Auclert et al. (2024), I show how the results of this paper can provide insight on this question.

Nodes in this framework correspond to time periods, and interactions between nodes are described by the matrix of intertemporal marginal propensities to consume (MPC): the spending in each period from an income shock in some period. For the comparative statics (or more precisely, the impulse response) of GDP on government expenditure — the fiscal multiplier — assumption 1 is satisfied under the condition that all MPCs are strictly positive.[33] Focusing on the impact fiscal multiplier, the effect on GDP today from deficit-financed spending today, proposition 5 can be applied to give a lower bound. This bound is potentially useful as it requires only a fraction of the intertemporal MPCs to enumerate — an object that little is known about empirically — while the exact value of the fiscal multiplier requires knowledge of the entire MPC matrix. Moreover, the bound produces sufficient conditions for the multiplier to be greater than one.

The equations of state in this framework derive from the market clearing equations at each time period $i \in \mathcal{N}$

$$Y_i = C_i\left(\boldsymbol{Y} - \boldsymbol{T}\right) + G_i \tag{93}$$

where Y_i is GDP (the endogenous state), G_i is government expenditure (the exogenous shock), and T_i is taxation. $C_i\left(\cdot\right)$ is private consumption in period i, which is a function of after-tax income in all time periods $\boldsymbol{Y} - \boldsymbol{T}$, as agents are able to save and borrow. In Auclert et al. (2024), the model underlying equation (93) is infinite horizon and therefore the domain of i is $\{0, 1, ..., \infty\}$. However the results of the current paper require a finite domain, therefore I consider a truncation, $i \in \mathcal{N} = \{0, ..., N\}$, for some large N. This is mostly innocuous in practical terms, as Auclert et al. (2024) also truncate when numerically solving the model.

A key object in characterizing the effect of fiscal policy is the intertemporal marginal propensity to consume, $M_{ij} \equiv \frac{\partial C_i}{\partial Y_j}$. This describes the change in consumption in period i due to an income increase in period j. The government can save and

[33] As long as the model can be mapped into the form given in section 2, the results also apply to impulse responses.

borrow and therefore its fiscal policy is subject to a lifetime budget constraint

$$\sum_{i\in\mathcal{N}} \frac{G_i}{(1+r)^i} = \sum_{i\in\mathcal{N}} \frac{T_i}{(1+r)^i} \tag{94}$$

where $r \geq 0$ is the interest rate. I will focus on the comparative static corresponding to the impact fiscal multiplier, $\frac{\partial Y_0}{\partial G_0}$: the change in GDP today, $\mathrm{d}Y_0$, due to the government spending today, $\mathrm{d}G_0 > 0$ (with $\mathrm{d}G_i = 0$ for $i > 0$), that is deficit-financed through an arbitrary tax schedule in the future, $\{\mathrm{d}T_i\}_{i>0}$.

If one assumes: 1) $\forall i, j : M_{ij} > 0$, and 2) there exists an $n \geq 1$ such that $\mathrm{d}Y_n \geq \mathrm{d}T_n$, the results in this paper imply (proof provided at the end of this section)

$$\frac{\partial Y_0}{\partial G_0} \geq \frac{1 - \sum_{j\in\mathcal{N}\setminus\{0,n\}} \delta_j s_j^T}{1 - M_{00}} \tag{95}$$

where

$$s_j^T \equiv \frac{(1+r)^{-j}\,\mathrm{d}T_j}{\sum_{j\in\mathcal{N}\setminus 0} (1+r)^{-i}\,\mathrm{d}T_i} \tag{96}$$

is the taxation share coming from period j (in present value terms) and

$$\delta_j = 1 - \frac{1}{(1+r)^{n-j}} \frac{M_{nj}}{1 - M_{jj}}$$

is the iDD degree in this model. Assumption 1) implies that all intertemporal MPCs are strictly positive. This rules out inferior income effects, which isn't very restrictive given this is the MPC for aggregate consumption, and assumes there is some transmission of income between all periods. This assumption is satisfied in all the applications considered in Auclert et al. (2024) (see their footnote A-14). Assumption 2) implies there is a time period where the GDP response is greater than the taxation levied in this period. This would likely by true if we set $n = 1$ while having most of the taxation during periods $j > 1$. Or if we only tax up to period $n - 1$ for some $n > 1$, as then the restriction reduces to $\mathrm{d}Y_n \geq 0$, and we know GDP impulse responses from fiscal spending shocks are usually nonnegative empirically (see e.g. Ramey and Zubairy 2018).

The lower bound of the impact multiplier given by equation (95) can be intuitively understood as follows. The $\frac{1}{1-M_{00}}$ component is the "static" Keynesian cross analogue: spending in period zero implies output will increase by one over one minus the static

MPC, M_{00}. However, the static Keynesian cross ignores the impact of the future taxation on today's output. The term $-\sum_{j\in\mathcal{N}\setminus\{0,n\}}\delta_j s_j^T$ is a sufficient statistic for a bound on this, reflecting the reduction in output today due to the negative income effect of taxation in the future. The iDD degree δ_j is bounding the effect from each future period j, and this is weighted by the taxation share in each period, s_j^T.

The iDD degree δ_j is smaller, and so the lower bound on the impact multiplier higher, if M_{jj} or M_{nj} are greater, as this corresponds to more of the negative income effect of the tax in period j being offset by consumption reduction in period j or n, respectively, as opposed to reducing consumption and therefore output, Y_0, now.

A key feature of the lower bound of the impact multiplier in equation (95) is that it requires substantially less information to enumerate than the point value of the impact multiplier. To illustrate the potential value of this, suppose the government spending is financed entirely by taxation in period 1, $\mathrm{d}T_1 = (1+r)\,\mathrm{d}G_0$, and choose $n = 2$. Then, the bound becomes

$$\frac{\partial Y_0}{\partial G_0} \geq \frac{1-\delta_1}{1-M_{00}} = \frac{M_{21}}{(1+r)\,(1-M_{00})\,(1-M_{11})} \tag{97}$$

Only three elements of the MPC matrix are required to enumerate the bound, M_{00}, M_{11}, M_{21}, while the entire $N \times N$ matrix is required for the point value. As Auclert et al. (2024) explain, we have very little empirical evidence on most of the elements of MPC matrix (typically limited to the first few elements of the first column, $M_{i,0}$), and those authors resort to using a structural model to fill in the gaps. The bound dramatically reduces this burden.

To demonstrate, suppose one assumes temporal symmetry such that $M_{11} = M_{00}$ and $M_{21} = M_{10}$,[34] which facilitates calibration to empirical estimates as these elements are known: $M_{00} = 0.51, M_{10} = 0.18$ (Fagereng et al., 2021), and set $r = 0.05$. The multiplier lower bound implied by equation (97) is then 0.71. This magnitude is non-trivial, and implies that the point value of the impact multiplier is fairly large regardless of, for instance, the structural model one might use to set the remaining elements of the MPC matrix.

The requirement of less information also allows one to derive sufficient conditions on the static MPC, M_{00}, such that the multiplier is greater one, $\frac{\partial Y_0}{\partial G_0} \geq 1$. Equation

[34] If this symmetry holds for all elements of the matrix, M is a Toeplitz matrix, which is approximately the case (quasi-Toeplitz) for many modern macroeconomic models (Auclert et al., 2023).

(97) immediately reveals a sufficient condition for this: $M_{00} \geq \delta_1$. Intuitively, the positive income effect on GDP today from the static MPC, M_{00}, is sufficiently greater than the negative income effect on GDP today from taxation tomorrow, δ_1 (recalling that the iDD degree bounds this). Moreover, if one continues with assuming the aforementioned temporal symmetry, and sets $M_{10} = 0.18, r = 0.05$, then the lower bound on the static MPC such that the fiscal multiplier is greater than one is $M_{00} \geq 1 - \sqrt{\frac{M_{10}}{1+r}} = 0.59$. This is very close to the empirical value of 0.51, implying that above-unity impact fiscal multipliers would be guaranteed in this framework if static MPCs are only slightly higher than those estimated.

Proof. Of Equation (95). Equation (93) forms the basis of the equations of state corresponding to equation (1). However, a simplification in the comparative static bounds can be achieved by not directly using this, instead by considering it first in differential form. Totally differentiating equation (93) gives

$$\begin{aligned} i \in \mathcal{N}: \quad 0 &= \mathrm{d}Y_i - \sum_{j\in\mathcal{N}} M_{ij}\left(\mathrm{d}Y_j - \mathrm{d}T_j\right) - \mathrm{d}G_i \\ &= \left(\mathrm{d}Y_i - \mathrm{d}T_i\right) - \sum_{j\in\mathcal{N}} M_{ij}\left(\mathrm{d}Y_j - \mathrm{d}T_j\right) - \left(\mathrm{d}G_i - \mathrm{d}T_i\right) \qquad (98) \\ &= \sum_{j\in\mathcal{N}} \left(I - M\right)_{ij}\left(\mathrm{d}Y_j - \mathrm{d}T_j\right) - \left(\mathrm{d}G_i - \mathrm{d}T_i\right) \qquad (99) \end{aligned}$$

Solving for $\mathrm{d}Y_j$ cannot proceed directly because $I - M$ is singular; equivalently, the solutions $\mathrm{d}Y_j$ to equation (99) are underdetermined. This is because M obeys the following in order to satisfy lifetime budget constraints

$$j \in \mathcal{N}: \quad \sum_{i\in\mathcal{N}} \frac{M_{ij}}{(1+r)^i} = \frac{1}{(1+r)^j} \tag{100}$$

Because $I - M$ is singular, there exists an eigenvector $\boldsymbol{v}$ of $I - M$ with eigenvalue zero. This means we can rewrite equation (99) without loss of generality as follows

$$i \in \mathcal{N}: \quad 0 = \sum_{j\in\mathcal{N}} \left(I - M\right)_{ij}\left(\mathrm{d}Y_j - \mathrm{d}T_j - av_j\right) - \left(\mathrm{d}G_i - \mathrm{d}T_i\right) \tag{101}$$

for any constant $a \in \mathbb{R}$, which reflects the indeterminacy. Essentially, the level of the solutions $\mathrm{d}Y_j$ is not determined. The model only determines the relative values $\mathrm{d}Y_j$, and these are solved for by setting $\mathrm{d}Y_j - \mathrm{d}T_j - av_j = 0$ for one j, and using only $N-1$

of the i equations from equation (101). I set $\mathrm{d}Y_n - \mathrm{d}T_n - av_n = 0$ and drop the $i = n$ equation, giving

$$i \in \mathcal{N}\backslash n: \quad 0 = \sum_{j\in\mathcal{N}\backslash n} (I-M)_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j - (\mathrm{d}Y_n - \mathrm{d}T_n)\frac{v_j}{v_n}\right) - (\mathrm{d}G_i - \mathrm{d}T_i) \tag{102}$$

noting that $v_n > 0$ ($\boldsymbol{v}$ is also an eigenvector of M with eigenvalue 1, and because $M > 0$, then $\boldsymbol{v} > 0$ by the Perron Theorem). The Jacobian of equation (102) is $\{I - M\}_{i,j\in\mathcal{N}\backslash n}$. This matrix does not satisfy assumption 1 in general, however it is generalized diagonally dominant (remark 4): multiply each i equation by $(1+r)^{-i}$

$$i \in \mathcal{N}\backslash n: \quad 0 = \mathrm{d}f\left(\tilde{\boldsymbol{Y}}, G_0\right) \equiv \sum_{j\in\mathcal{N}\backslash n} \frac{(I-M)_{ij}}{(1+r)^i} \underbrace{\left(\mathrm{d}Y_j - \mathrm{d}T_j - (\mathrm{d}Y_n - \mathrm{d}T_n)\frac{v_j}{v_n}\right)}_{\equiv \mathrm{d}\tilde{Y}_j} - \frac{\mathrm{d}G_i - \mathrm{d}T_i}{(1+r)^i} \tag{103}$$

Impose some tax policy $T_i = T_i(G_0)$ that satisfies budget balance equation (94)

$$G_0 = \sum_{i\in\mathcal{N}\backslash 0} \frac{T_i}{(1+r)^i} \tag{104}$$

recalling that $\forall i > 0 : G_i = 0$. Then, equation (103) is the (differential of the) equations of state in the form I'll apply my bounds to, with $\tilde{Y}_j$ the endogenous state, and G_0 the exogenous shock. The Jacobian of this system is given by

$$i \in \mathcal{N}\backslash n, j \in \mathcal{N}\backslash n: \quad \nabla_{ij} \equiv \frac{\partial f_i}{\partial \tilde{Y}_j} = (1+r)^{-i}(I_{ij} - M_{ij}) \tag{105}$$

which satisfies assumption 1 because

$$\begin{aligned} i \in \mathcal{N}\backslash n: \quad \sum_{j\in\mathcal{N}\backslash\{i,n\}} |\nabla_{ji}| &= \sum_{j\in\mathcal{N}\backslash\{i,n\}} (1+r)^{-j} M_{ji} \\ &= \sum_{j\in\mathcal{N}} (1+r)^{-j} M_{ji} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni} \\ &= (1+r)^{-i} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni} \\ &= |\nabla_{ii}| - (1+r)^{-n} M_{ni} \\ &< |\nabla_{ii}| \end{aligned} \tag{106}$$

where line three used equation (100), line four used $\nabla_{ii} = (1+r)^{-i}(1-M_{ii}) \geq 0$ by equation (100) and $M > 0$, and the last line used $M > 0$. The iDD degrees are given by

$$
\begin{aligned}
i \in \mathcal{N}\backslash n: \quad \delta_i &= \frac{\sum_{j\in\mathcal{N}\backslash\{i,n\}} |\nabla_{ji}|}{|\nabla_{ii}|} \\
&= \frac{\sum_{j\in\mathcal{N}\backslash\{i,n\}} (1+r)^{-j} M_{ji}}{(1+r)^{-i}(1-M_{ii})} \\
&= \frac{\sum_{j\in\mathcal{N}} (1+r)^{-j} M_{ji} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni}}{(1+r)^{-i}(1-M_{ii})} \\
&= \frac{(1+r)^{-i} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni}}{(1+r)^{-i}(1-M_{ii})} \\
&= 1 - \frac{(1+r)^{-n} M_{ni}}{(1+r)^{-i}(1-M_{ii})}
\end{aligned}
$$

where the second line used $\nabla_{ii} \geq 0$; the fourth line used equation (100). The direct effects are given by[35]

$$
\begin{aligned}
\frac{\partial f_i}{\partial G_0} &= -\frac{1}{\mathrm{d}G_0} \frac{\mathrm{d}G_i - \mathrm{d}T_i}{(1+r)^i} \\
&= -\frac{I_{i0} - \frac{\mathrm{d}T_i}{\mathrm{d}G_0}}{(1+r)^i} \\
&= -(1+r)^i I_{i0} + s_i^T
\end{aligned} \tag{107}
$$

The last line follows because $\frac{\mathrm{d}T_i}{\mathrm{d}G_0} = \frac{(1+r)^i s_i^T \sum_{j\in\mathcal{N}\backslash 0}(1+r)^{-j}\mathrm{d}T_j}{\mathrm{d}G_0} = (1+r)^i s_i^T$, which used the definition of s_i^T equation (96) in the first equality, and that $\sum_{j\in\mathcal{N}\backslash 0}(1+r)^{-j}\mathrm{d}T_j = \mathrm{d}G_0$ by the budget balance equation (104) in the second equality. Thus, the lower

[35]Note that the $\mathrm{d}T_n$ term in equation (103) is part of $\mathrm{d}\tilde{Y}$ and therefore is not part of the direct effect.

bound on the impact multiplier is derived by

$$\begin{aligned}
\frac{\mathrm{d}Y_0}{\mathrm{d}G_0} &= \frac{\mathrm{d}\left(Y_0 - T_0\right)}{\mathrm{d}G_0} \\
&\geq \frac{\mathrm{d}\left(Y_0 - T_0\right) - \mathrm{d}\left(Y_n - T_n\right)\frac{v_j}{v_n}}{\mathrm{d}G_0} \\
&= \frac{\mathrm{d}\tilde{Y}_0}{\mathrm{d}G_0} && (108) \\
&\geq \frac{1}{\left|\nabla_{00}\right|}\left(\frac{\partial f_0}{\partial G_0} - \sum_{j\in\mathcal{N}\backslash 0}\delta_j\left|\frac{\partial f_j}{\partial G_0}\right|\right) && (109) \\
&\geq \frac{1}{1-M_{00}}\left(1 - \sum_{j\in\mathcal{N}\backslash 0}\delta_j s_j^T\right) && (110)
\end{aligned}$$

where the first line used $T_0 = 0$ by assumption; the second line used $\mathrm{d}Y_n \geq \mathrm{d}T_n$ by assumption; the third line used the definition of $\tilde{Y}_0$. The fourth line applied proposition 5 to $\frac{\mathrm{d}\tilde{Y}_0}{\mathrm{d}G_0}$ based on equations of state equation (103), noting that both sign conditions, equations (73) and (74), are satisfied with $s_1 = 1, s_2 = -1$. The sign of the comparative static is determined because equation (46) is satisfied, $\sum_{i\in\mathcal{N}\backslash 0}\delta_i\left|\frac{\partial f_i}{\partial G_0}\right| \leq \delta_{-0}\sum_{i\in\mathcal{N}\backslash 0}s_i^T = \delta_{-0} < 1 = \left|\frac{\partial f_0}{\partial G_0}\right|$, implying a positive comparative static $-\mathrm{sgn}\left(\nabla_{ii}\right)\mathrm{sgn}\left(\frac{\partial f_0}{\partial G_0}\right) = -\mathrm{sgn}\left(1-M_{00}\right)\mathrm{sgn}\left(-1\right) = 1$. The lower bound from equation (75) then implies equation (109). The last line, equation (110) follows using equations (105) and (107). $\square$

C.3 International Trade and Economic Geography

Consider the framework in Allen et al. (2020) that encapsulates many of the workhorse models in the international trade and economic geography literature. Notably it is a strict generalization of Arkolakis et al. (2012) under their CES demand assumption R3', which corresponds to the case where the scale elasticity (introduced below) is $\psi = 0$ (see footnote 12 in Arkolakis et al. 2012).

Nodes in this framework correspond to locations and interactions between nodes are described by the bilateral trade flows between locations, along with the demand and supply elasticities. For comparative statics on production prices, I show assumption 1 is satisfied under precisely one of the conditions the authors give for equilibrium uniqueness: the demand and supply elasticities are each greater than $-\frac{1}{2}$, and trade costs are quasi-symmetric. The bounds for the comparative statics (proposition 2) of either price or the real wage with respect to trade costs depend only on the demand and supply elasticities, the own trade share, and the trade share with the corresponding location directly impacted by the trade cost shock. I draw a parallel of these sufficient statistics to the Arkolakis et al. (2012) welfare formula.

There are $i \in \mathcal{N}, j \in \mathcal{N}$ locations, in which an aggregate good is traded across locations subject to iceberg trade frictions, $\tau_{ij} > 0$. Consumers have CES preferences, implying demand from consumers in location j for products produced in location i is $X_{ij} = \left(\frac{p_i \tau_{ij}}{P_j}\right)^{-\phi} E_j$, where p_i is the production price for products produced in location i, $P_j^{-\phi} = \sum_{i \in \mathcal{N}} \tau_{ij}^{-\phi} p_i^{-\phi}$ the consumer price index in j, E_j is total expenditure by consumers in j, and $\phi \in \mathbb{R}$ is the demand elasticity. Quantity supplied by location i is given by $Q_i = \kappa \bar{c}_i \left(\frac{p_i}{P_i}\right)^{\psi}$, where $\psi \in \mathbb{R}$ is the supply elasticity, $\bar{c}_i$ is productivity, and $\kappa > 0$ is a (possibly endogenous) scalar. The output market clears and trade is balanced up to an exogenous deficit.

A notable case considered by the authors is the one of balanced trade and quasi-symmetric trade costs,

$$\forall i \in \mathcal{N}, j \in \mathcal{N}: \quad \tau_{ij} = \tau_i^A \tau_j^B \tilde{\tau}_{ij} \tag{111}$$

where $\tau_i^A > 0, \tau_i^B > 0$ and $\tilde{\tau}_{ij} = \tilde{\tau}_{ji} > 0$. Under this assumption, the production price solves the following set of equations (see section A2.3 of Allen et al., 2020) for all $i \in \mathcal{N}$

$$0 = f_i\left(\ln \mathbf{p}, \ln \tilde{\boldsymbol{\tau}}\right) \equiv \underbrace{\kappa p_i^{1+\psi+\psi\frac{1+\psi+\phi}{\phi-\psi}} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{\psi\phi}{\phi-\psi}} \bar{c}_i^{\frac{\phi}{\psi-\phi}}}_{=Y_i} - \sum_{j \in \mathcal{N}} \underbrace{\kappa \tilde{\tau}_{ij}^{-\phi} p_i^{-\phi} \left(\tau_i^A \tau_j^A\right)^{-\phi} p_j^{-\phi}}_{=X_{ij}} \tag{112}$$

where $Y_i \equiv p_i Q_i$ is dollar output. These form the equations of state of the current paper, equation (1), with the (log of) production price, $\ln p_i$, as the endogenous state and the (symmetric component of the) trade costs, $\ln \tilde{\tau}_{ij}$, as the exogenous shock. Using proposition 2, I derive the bounds for the comparative statics

$$\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}} = -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial \ln \tilde{\tau}_{lj}} \tag{113}$$

for any $i \in \mathcal{N}, l \in \mathcal{N}, j \in \mathcal{N}$. The Jacobian is

$$\nabla_{ij} \equiv \frac{\partial f_i(\mathbf{p}, \tilde{\boldsymbol{\tau}})}{\partial \ln p_j} = \phi \frac{1+\psi+\phi}{\phi-\psi}\left(Y_i I_{ij} + \frac{\phi-\psi}{1+\psi+\phi} X_{ij}\right) \tag{114}$$

and the direct effects matrix

$$\frac{\partial f_i}{\partial \ln \tilde{\tau}_{kj}} \equiv \frac{\partial f_i(\mathbf{p}, \tilde{\boldsymbol{\tau}})}{\partial \ln \tilde{\tau}_{kj}} = 2\phi X_{ij} I_{ki}, \quad \left|\frac{\partial f_{-i}}{\partial \ln \tilde{\tau}_{kj}}\right| = 2\left|\phi\right| \sum_{l \in \mathcal{N} \setminus i} X_{lj} I_{ki} \tag{115}$$

Hat algebra has been used to write both expressions in terms of observable objects (Y, X) rather than unobservable objects $(\tau, \bar{c})$ — see Allen et al. (2020) for details, and see Dekle et al. (2008) for the seminal paper on this. A sufficient condition for the Jacobian to satisfy assumption 1 is

$$\{\phi > -0.5,\ \psi > -0.5\} \text{ or } \{\phi < -0.5,\ \psi < -0.5\} \tag{116}$$

Allen et al. (2020) theorem 1.iii shows that equation (116) with quasi-symmetric trade costs is sufficient for the model to have a unique interior equilibrium. To see that equation (116) implies column diagonal dominance of the Jacobian, assumption 1,[36]

$$\begin{aligned} |\nabla_{ii}| - \sum_{i \neq j} |\nabla_{ij}| &= \left|\phi \frac{1+\psi+\phi}{\phi-\psi}\right| \left[\left(1 - \left|\frac{\phi-\psi}{1+\psi+\phi}\right|\right) Y_i + \left(\frac{\phi-\psi}{1+\psi+\phi} + \left|\frac{\phi-\psi}{1+\psi+\phi}\right|\right) X_{ii}\right] \\ &> 0 \end{aligned}$$

where I used that $\left|\frac{\phi-\psi}{1+\psi+\phi}\right| < 1$ under equation (116). This last fact can be confirmed by showing $|1+\phi+\psi| - |\phi-\psi| > 0$. If $\phi > -0.5, \psi > -0.5$, then

$$|1+\phi+\psi| - |\phi-\psi| = \begin{cases} \phi \geq \psi & 1+\phi+\psi-\phi+\psi = 1+2\psi > 0 \\ \phi < \psi & 1+\phi+\psi+\phi-\psi = 1+2\phi > 0 \end{cases}$$

[36] Allen et al., 2020 prove the Jacobian is row diagonally dominant under $\phi > 0$ and $\psi > 0$ in lemma 5.

and if $\phi < -0.5, \psi < -0.5$, then

$$|1+\phi+\psi| - |\phi-\psi| = \begin{cases} \phi \geq \psi & -(1+\phi+\psi) - \phi + \psi = -1 - 2\phi > 0 \\ \phi < \psi & -(1+\phi+\psi) + \phi - \psi = -1 - 2\psi > 0 \end{cases}$$

The iDD degree in this model is given by

$$\delta_i = \left| \frac{\phi-\psi}{1+\psi+\phi} \right| \frac{1 - \frac{X_{ii}}{Y_i}}{1 + \frac{\phi-\psi}{1+\psi+\phi} \frac{X_{ii}}{Y_i}}$$

Thus, assuming the condition in equation (116), we can apply proposition 2

$$\left| \frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \right| \in \frac{2\delta_i \frac{X_{ij}}{Y_i}}{1 - \frac{X_{ii}}{Y_i}} \left(\frac{1}{1+\delta_i}, \frac{1}{1-\delta_i} \right) \tag{117}$$

$$\operatorname{sgn}\left(\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \right) = -\operatorname{sgn}\left(\nabla_{ii}\right) \cdot \operatorname{sgn}\left(\frac{\partial f_i}{\partial \ln \tilde{\tau}_{ij}} \right) = \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi} \right) \tag{118}$$

$$l \neq i: \quad \left| \frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}} \right| < \frac{2\delta_i}{1 - \frac{X_{ii}}{Y_i}} \frac{\delta_l \frac{X_{lj}}{Y_i}}{1-\delta_i} \tag{119}$$

where I used that $|\nabla_{ii}| = \left| \phi \frac{1+\psi+\phi}{\phi-\psi} \right| Y_i \left(1 + \frac{\phi-\psi}{1+\psi+\phi} \frac{X_{ii}}{Y_i} \right) = |\phi| \, Y_i \frac{1-\frac{X_{ii}}{Y_i}}{\delta_i}$, and that equation (54) is satisfied only for the comparatives statics, $\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}}$, with $l = i$ because

$$\begin{aligned} \left| \frac{\partial f_i}{\partial \ln \tilde{\tau}_{lj}} \right| - \left| \frac{\partial f_{-i}}{\partial \ln \tilde{\tau}_{lj}} \right| &= 2\,|\phi| \left(X_{ij} I_{li} - \sum_{k \in \mathcal{N} \setminus i} X_{kj} I_{lk} \right) \\ &= \begin{cases} l = i & 2\,|\phi|\, X_{ij} \geq 0 \\ l \neq i & -2\,|\phi|\, X_{lj} \leq 0 \end{cases} \end{aligned}$$

The comparative static bounds in equations (117) and (119) are enumerated with knowledge of only the demand elasticity ϕ, the supply elasticity ψ, the own-trade share for i $\frac{X_{ii}}{Y_i}$, and the level of trade between the two countries directly affected by the trade cost, l, j, relative to output in i, $\frac{X_{lj}}{Y_i}$.

I can use the above bounds on prices to also bound welfare. For a range of canonical international trade models isomorphic to the Universal Gravity framework, the change in welfare of a worker in location i is $W_i = B_i \left(\frac{p_i}{P_i} \right)^{1+\psi}$, with B_i an exogenous scalar (Allen et al. 2020 table 1). The change in welfare with respect to a trade cost shock, $\tilde{\tau}_{ij}$, is then

$$\frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} = (1+\psi) \frac{\partial \ln \frac{p_i}{P_i}}{\partial \ln \tilde{\tau}_{ij}} = (1+\psi) \frac{2\phi+1}{\phi-\psi} \frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \tag{120}$$

where the second equality used $\frac{\partial \ln \frac{p_i}{P_i}}{\partial \ln \tilde{\tau}_{ij}} = \frac{2\phi+1}{\phi-\psi} \frac{\partial \ln p_i}{\partial \ln \tau_{ij}}$ from Allen et al. (2020). Using the bounds and sign for $\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}}$ from equations (117) and (118) gives

$$\left| \frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \right| = \left| \frac{(1+\psi)(2\phi+1)}{\phi-\psi} \right| \frac{2\delta_i \frac{X_{ij}}{Y_i}}{1-\frac{X_{ii}}{Y_i}} \left(\frac{1}{1+\delta_i}, \frac{1}{1-\delta_i} \right) \tag{121}$$

$$\operatorname{sgn}\left(\frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \right) = \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi} \frac{(1+\psi)(2\phi+1)}{\phi-\psi} \right) \tag{122}$$

In section 4.2, I consider a proportional increase in trade costs with the rest of the world $\forall j \neq i : \mathrm{d} \ln \tilde{\tau}_{ij} = \mathrm{d} \ln \overline{\tau}$. The change in welfare in i is

$$\begin{aligned}
\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} &= \sum_{j \in \mathcal{N} \setminus i} \frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \\
&= (1+\psi) \frac{2\phi+1}{\phi-\psi} \sum_{j \in \mathcal{N} \setminus i} \frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \\
&= -(1+\psi) \frac{2\phi+1}{\phi-\psi} \sum_{j \in \mathcal{N} \setminus i, k \in \mathcal{N}} \left\{ \nabla^{-1} \right\}_{ik} \frac{\partial f_k}{\partial \ln \tilde{\tau}_{ij}} \\
&= -(1+\psi) \frac{2\phi+1}{\phi-\psi} \sum_{j \in \mathcal{N} \setminus i, k \in \mathcal{N}} \left\{ \nabla^{-1} \right\}_{ik} 2\phi X_{kj} I_{ik} \\
&= -(1+\psi) \frac{2\phi+1}{\phi-\psi} \sum_{j \in \mathcal{N} \setminus i} \left\{ \nabla^{-1} \right\}_{ii} 2\phi X_{ij} \\
&= -(1+\psi) \frac{2\phi+1}{\phi-\psi} \left\{ \nabla^{-1} \right\}_{ii} 2\phi (Y_i - X_{ii})
\end{aligned}$$

The second line used equation (120); the third line used equation (113); and the fourth line used equation (115). In deriving equation (30) in the main text, I further set $\psi = 0$, which corresponds to the case of Arkolakis et al. (2012),

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -2(2\phi+1) \left\{ \nabla^{-1} \right\}_{ii} (Y_i - X_{ii}), \quad \nabla_{ij} = (1+\phi) \left(Y_i I_{ij} + \frac{\phi}{1+\phi} X_{ij} \right)$$

and use a transformed version of $\nabla_{ij} \to \frac{1}{(1+\phi)Y_i}\nabla_{ij}$ for notational convenience, giving equation (30). The bounds in equation (31) are derived by

$$
\begin{aligned}
\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} &= \sum_{j\in\mathcal{N}\setminus i} \frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \\
&= \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)\frac{(1+\psi)(2\phi+1)}{\phi-\psi}2\delta_i \sum_{j\in\mathcal{N}\setminus i}\frac{\frac{X_{ij}}{Y_i}}{1-\frac{X_{ii}}{Y_i}}\left(\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right) \\
&= \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)\frac{(1+\psi)(2\phi+1)}{\phi-\psi}2\delta_i\left(\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right) \\
&= -\frac{2\phi+1}{|\phi|}2\delta_i\left(\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right), \quad \delta_i = \left|\frac{\phi}{1+\phi}\right|\frac{1-\frac{X_{ii}}{Y_i}}{1+\frac{\phi}{1+\phi}\frac{X_{ii}}{Y_i}}
\end{aligned}
$$

where the second line used equations (121) and (122); and the last line set $\psi = 0$ (corresponding to Arkolakis et al. 2012). Equation (31) follows by imposing $\phi > 0$.

C.4 Industrial Organization: Oligopoly

Consider a canonical Bertrand oligopoly model with differentiated products from the Industrial Organization literature (Milgrom and Roberts, 1990). This field of study has attracted renewed attention due to the rising importance of market power and oligopolistic industries (De Loecker et al., 2020; Azar and Vives, 2021). My results hold particular use for emerging research permitting rich heterogeneity, often by exploiting a network-style approach (Bimpikis et al., 2019; Galeotti et al., 2024; Pellegrino, 2025).

Nodes in this framework correspond to products, and interactions between nodes are described by the Hessian of demand between products, and the mark-up. For the comparative statics on output prices, row diagonal dominance (remark 3) is implied by workhorse demand structures, such as CES, logit, linear (with products assumed substitutes), and translog (with products assumed substitutes plus an additional restriction on the parameters).[37] Milgrom and Roberts (1990) demonstrates this property and proves its sufficiency for equilibrium uniqueness. My results imply

[37] Diagonal dominance has been shown to be relevant in other market structures too. The framework in Dixit (1986) nests Bertrand, Cournot, market-share and competitive markets structures, and they appeal to diagonal dominance for stability of the equilibrium. Pellegrino (2025) consider Bertrand with a generalized hedonic-linear demand system; one can verify that their (inverse) demand satisfies diagonal dominance.

that this condition is also sufficient to bound the comparative statics. I use these to characterize price-cost passthrough in a general asymmetric oligopoly, extending the analysis of symmetric models in the literature (Weyl and Fabinger, 2013).

There are $i \in \mathcal{N}$ firms each producing one good with constant unit costs, c_i, and with demand $D_i(\boldsymbol{p})$, where $\boldsymbol{p} = \{p_i\}_{i\in\mathcal{N}}$ are the endogenous product prices. $D_i(\boldsymbol{p})$ is assumed to be twice-continuously differentiable. The log profits of firm i are

$$\pi_i(\boldsymbol{p}, c_i) = \ln\left[(p_i - c_i) D_i(\boldsymbol{p})\right]$$

and the firm chooses p_i to maximize $\pi_i(\boldsymbol{p}, c_i)$, taking all other firms' prices as given. The optimal price solves the first-order condition

$$\frac{\partial \pi_i}{\partial \ln p_i} = \underbrace{\frac{1}{1-\frac{c_i}{p_i}} + \frac{\partial \ln D_i(\boldsymbol{p})}{\partial \ln p_i}}_{\equiv f_i(\ln \boldsymbol{p}, \ln \mathbf{c})} = 0 \tag{123}$$

Equation (123) corresponds to the equations of state in my framework, equation (1), with $\ln \boldsymbol{p}$ the endogenous state variable, and $\ln \mathbf{c}$ the exogenous shock. The Jacobian, accordingly, is,

$$\nabla_{ij} \equiv \frac{\partial f_i(\ln \boldsymbol{p}, \ln \mathbf{c})}{\partial \ln p_j} = -\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij} + \frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j} \tag{124}$$

The second order condition is

$$0 > \frac{\partial^2 \pi_i}{\partial \ln p_i^2} = \nabla_{ii} \tag{125}$$

which implies the diagonal of the Jacobian is negative. The direct effect matrix is

$$\frac{\partial f_i}{\partial \ln c_j} \equiv \frac{\partial f_i(\ln \boldsymbol{p}, \ln \mathbf{c})}{\partial \ln c_j} = \frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij}$$

The pass-through to p_i of a change in c_i is given by

$$\frac{\partial p_i}{\partial c_i} = \left\{\nabla^{-1}\right\}_{ii} \frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}$$

In equation (32) in the main text, I use a transformed version of $\nabla_{ij} \rightarrow \frac{\left(1-\frac{c_i}{p_i}\right)^2}{\frac{c_i}{p_i}} \nabla_{ij}$ for notational convenience. In deriving the bounds on this comparative static, I use proposition 3. The direct effects are only non-zero for the node shocked, $\forall i, j \neq i : \frac{\partial f_i}{\partial \ln c_j} = 0$, as required by equation (59), and the Jacobian is row diagonally dominant, as required by equation (58), across many standard demand structures for $D_i(\boldsymbol{p})$, as explained by Milgrom and Roberts (1990) pg 1271. Specifically: CES and logit; linear under substitutes; translog under substitutes and an additional parameter restriction.[38]

For example, CES demand is $D_i(\boldsymbol{p}) = y p_n^{r-1} / \sum_j p_j^r$, where $1 - r \geq 0$ is the elasticity substitution, and $y = \sum_i p_i D_i(\boldsymbol{p})$ is total expenditure. The Hessian of demand is

$$\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j} = r^2 \frac{p_i D_i}{y} \left(\frac{p_j D_j}{y} - I_{ij} \right)$$

thus

$$\begin{aligned} \sum_{j \notin \mathcal{N}} |\nabla_{ij}| &= \sum_{j \notin \mathcal{N}} r^2 \frac{p_i D_i}{y} \frac{p_j D_j}{y} = r^2 \frac{p_i D_i}{y} \left(1 - \frac{p_i D_i}{y}\right) \\ &< \frac{\frac{c_i}{p_i}}{\left(1 - \frac{c_i}{p_i}\right)^2} + r^2 \frac{p_i D_i}{y} \left(1 - \frac{p_i D_i}{y}\right) = |\nabla_{ii}| \end{aligned}$$

hence row diagonal dominance, equation (58), is always satisfied.

An alternative sufficient condition for row diagonal dominance is the case considered in the main text. Assume the Hessian of log demand is row diagonally dominant,

$$\forall i \in \mathcal{N} : \quad \left| \frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i} \right| > \sum_{j \in \mathcal{N} \setminus i} \left| \frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j} \right| \tag{126}$$

and the diagonal of the Hessian is positive

$$\forall i \in \mathcal{N} : \quad \frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i} > 0 \tag{127}$$

which implies demand is log-convex (as the Hessian of log-demand is positive semidefinite). For the equilibrium to be stable under these assumptions, one must restrict

[38] Logit requires taking the equations of state to be $0 = \frac{\partial \pi_i}{\partial p_i}$, as opposed to $0 = \frac{\partial \pi_i}{\partial \ln p_i}$. This is an application of remark 4, with $g_i' = \frac{1}{p_i}$.

the level of convexity. A sufficient condition is[39]

$$\frac{\frac{c_i}{p_i}}{2\left(1-\frac{c_i}{p_i}\right)^2} \geq \frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_i} \tag{128}$$

i.e. limiting how positive the diagonal of the Hessian can be. Together, equations (126), (127), and (128) are sufficient for row diagonal dominance of the Jacobian

$$\begin{aligned}
\left|\nabla_{ii}\right| - \sum_{j\in\mathcal{N}\setminus i}\left|\nabla_{ij}\right| &= \left|-\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij} + \frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_i}\right| - \sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_j}\right| \\
&> \left|-\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij} + \frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_i}\right| - \left|\frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_i}\right| \\
&> \frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij} - 2\frac{\partial^2 \ln D_i\left(\boldsymbol{p}\right)}{\partial \ln p_i \partial \ln p_i} \\
&> 0
\end{aligned}$$

The second line used equation (126), the third line used equations (127) and (128), and the fourth line used equation (128).

Turning to the comparative static bounds. For the general case, the bounds for $j = i$ from proposition 3 are

$$\frac{\partial p_i}{\partial c_i} \in \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\left|\nabla_{ii}\right|}\left[\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{1-\delta_i\delta_{-i}}\right], \qquad \delta_i = \frac{\sum_{j\in\mathcal{N}\setminus i}\left|\nabla_{ij}\right|}{\left|\nabla_{ii}\right|} \tag{129}$$

noting that the iDD degree is constructed using the row sum as we are using proposition 3. In the main text, I am interested in the lower bound, which can be manipulated

[39] An equilibrium in this model is considered stable if the Jacobian is negative semidefinite (see Dixit, 1986 pg 117), which is implied by equation (125) and diagonal dominance of ∇ (the latter being implied by equations 126, 127, and 128)

further using $\delta_{-i} < 1$

$$
\begin{aligned}
\frac{\partial p_i}{\partial c_i} &> \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{|\nabla_{ii}|}\frac{1}{1+\delta_i} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\sum_{j\in\mathcal{N}}|\nabla_{ij}|} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\left|-\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}+\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i}\right| + \sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j}\right|} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}-\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i} + \sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_j}\right|}
\end{aligned} \tag{130}
$$

where the second line used the definition of δ_i from equation (129); the third line used the definition of the Jacobian from equation (124); and the fourth line used the second-order condition equation (125).

The bounds under the case considered in the main text become

$$
\frac{\partial p_i}{\partial c_i} > \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}-\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i} + \left|\frac{\partial^2 \ln D_i(\boldsymbol{p})}{\partial \ln p_i \partial \ln p_i}\right|} > 1
$$

where the first inequality imposed equation (126) in equation (130), and the second inequality used equation (127). Thus, pass-through is greater than unity.

C.5 Network Games

Consider a broad class of games played on networks (Galeotti et al., 2010). These game theoretic models capture a wide variety of economic settings, such as peer effects, public goods and technology adoption. I consider the workhorse simultaneous-move game with unconstrained actions and linear best replies, as reviewed in chapter five of Bramoullé et al. (2016).

Nodes in this framework correspond to players, which may be interpreted as, for example, students or firms, depending on the application. Interactions are described by the network adjacency matrix (describing, for example, a social network), and the peer effect parameter (which modulates the strength and sign of the interactions). For the comparative static of a player's effort, I show that the Jacobian satisfies generalized diagonal dominance (remark 4), row diagonal dominance (remark 3) and a variant on signed diagonal dominance (remark 5) under assumptions that are regularly appealed to in the literature concerning equilibrium uniqueness, learning, and strategic complementarities. The bounds on the comparative static of a player's effort with respect to their own private benefit is identified solely from summary statistics of the adjacency matrix, such as the spectral radius, eigenvector centralities, or network degrees, which all have independent interest in the literature. As I show, these results are particularly useful for identifying the peer effect parameter in settings of incomplete network data, which is the empirically typical case (Lewis and Chandrasekhar, 2011).

Player $i \in \mathcal{N}$ choses effort $y_i \in \mathbb{R}$ to maximize a linear-quadratic utility function

$$u_i = \beta x_i y_i - \frac{1}{2} y_i^2 + \gamma \sum_{j \in \mathcal{N}} G_{ij} y_i x_j + \phi \sum_{j \in \mathcal{N}} G_{ij} y_i y_j$$

$G_{ij} \in \mathbb{R}$ is a signed, directed and weighted adjacency matrix, whose magnitude indicates the strength of the interaction between agents i and j. I assume $\forall i : G_{ii} = 0$ as standard (there is no interaction with oneself). $x_i \in \mathbb{R}$ is an exogenous characteristic of individual i. The utility value of exerting effort exhibits diminishing marginal utility and depends directly on i's own characteristic x_i with coefficient $\beta \in \mathbb{R}$, on its peers' characteristics, $\sum_{j \in \mathcal{N}} G_{ij} x_i$, with coefficient $\gamma \in \mathbb{R}$, and on it's peers' actions, $\sum_{j \in \mathcal{N}} G_{ij} y_i$, with coefficient $\phi \in \mathbb{R}$. The last term gives rise to strategic interactions between players when maximizing utility. ϕ scales the magnitude of the strategic interactions, and will be referred to as the peer effect parameter (also known as the payoff impact parameter). i and j's actions are strategic complements when $\phi G_{ij} \geq 0$, and strategic substitutes when $\phi G_{ij} \leq 0$.

The first order condition of $\max_{y_i} u_i$ yields player i's best reply

$$y_i = \beta x_i + \gamma \sum_{j \in \mathcal{N}} G_{ij} x_j + \phi \sum_{j \in \mathcal{N}} G_{ij} y_j \tag{131}$$

The second and third terms correspond to contextual and endogenous peer effects, in the language of Manski (1993). The Nash equilibrium is the $\boldsymbol{y}$ such that all players' best replies are satisfied, and these form the basis of the equations of state, equation (1). Consider the comparative static of effort with respect to the individual characteristic, $\frac{\partial y_i}{\partial x_j}$. In order to facilitate application of the bounds, first make a transformation of equation (131) as follows

$$y_i + \frac{\gamma}{\phi} x_i = \phi \sum_{j \in \mathcal{N}} G_{ij} \left(y_j + \frac{\gamma}{\phi} x_j \right) + \left(\beta + \frac{\gamma}{\phi} \right) x_i$$

and consider $\tilde{y}_i \equiv y_i + \frac{\gamma}{\phi} x_i$ as the endogenous state, with x_i the exogenous shock. The equations of state are then

$$0 = f_i \left(\tilde{\boldsymbol{y}}, \boldsymbol{x} \right) \equiv \tilde{y}_i - \phi \sum_{j \in \mathcal{N}} G_{ij} \tilde{y}_j - \left(\beta + \frac{\gamma}{\phi} \right) x_i \tag{132}$$

The corresponding Jacobian and direct effects are

$$\nabla_{ij} \equiv \frac{\partial f_i}{\partial \tilde{y}_j} = I_{ij} - \phi G_{ij}, \quad \frac{\partial f_i}{\partial x_j} = - \left(\beta + \frac{\gamma}{\phi} \right) I_{ij}$$

The reason for the aforementioned transformation is that the direct effect matrix $\frac{\partial f_i}{\partial x_j}$ of this system satisfies equation (59), a necessary condition for proposition 3.[40] The comparative static of interest, $\frac{\partial y_i}{\partial x_j}$, is related to this transformed system by

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial \tilde{y}_i}{\partial x_j} - \frac{\gamma}{\phi} I_{ij} \tag{133}$$

$$= \left\{ (I - \phi G)^{-1} \right\}_{ij} \left(\beta + \frac{\gamma}{\phi} \right) - \frac{\gamma}{\phi} I_{ij} \tag{134}$$

where the first line used $y_i = \tilde{y}_i - \frac{\gamma}{\phi} x_i$, and the second line used equation (3) for the system given in equation (132). The comparative static bounds will be applied directly to $\frac{\partial \tilde{y}_i}{\partial x_j}$, which will be used to bound $\frac{\partial y_i}{\partial x_j}$ through equation (133).

Diagonal dominance is satisfied under conditions typically invoked for equilibrium uniqueness. As outlined in Bramoullé et al. (2016) section 5.4.1 (see also Bramoullé

[40] Whereas the direct effect matrix of the untransformed system would be $-\beta I_{ij} - \gamma \sum_{j \in \mathcal{N}} G_{ij}$, which does not satisfy equation (59) in general.

et al. 2014), a Nash equilibrium exists and is unique if $I - \phi G$ is invertible, and is asymptotically stable if the spectral radius of ϕG is less than one. A sufficient condition for both of these is $|\phi|\,\mu < 1$, where μ is the spectral radius of $|G|$.[41] If one also assumes G is irreducible, which is equivalent to the often invoked assumption of the network being is strongly connected,[42] in addition to $|\phi|\,\mu < 1$, then the conditions required for proposition 4 are satisfied (see remark 4).[43] The proposition gives the following bounds for the comparative statics, shown for the case where $\beta + \frac{\gamma}{\phi} \geq 0$ (the bracketed term in equation 135 is flipped when $\beta + \frac{\gamma}{\phi} < 0$)

$$\forall i \in \mathcal{N}: \quad \frac{\partial y_i}{\partial x_i} \in \left[\frac{\beta + \frac{\gamma}{\phi}}{1 + \phi^2\mu^2}, \frac{\beta + \frac{\gamma}{\phi}}{1 - \phi^2\mu^2}\right] - \frac{\gamma}{\phi} \tag{135}$$

$$\forall i \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\frac{\partial y_i}{\partial x_j}\right| \leq \frac{|\phi|\,\mu\left|\beta + \frac{\gamma}{\phi}\right|\frac{v_j}{v_i}}{1 - \phi^2\mu^2} \tag{136}$$

where $\{v_i\}$ is the eigenvector of $|G|$ with eigenvalue μ. The bracketed term in equation (135), and the expression in equation (136), use the bounds and sign from equations (69), (70) and (71), applied to $\frac{\partial \tilde{y}_i}{\partial x_j}$, noting that equation (68) is satisfied for $j = i$. The bounds only depend on low dimensional summary statistics of the adjacency matrix G, which have independent interest in the literature: the spectral radius μ (Ballester et al., 2006; Bramoullé et al., 2014) and the eigenvectors v_i, which correspond to the eigenvector centralities of each node (Golub and Jackson, 2010).

I next consider a number of special cases. Suppose that $G \geq 0$ (element-wise non-negativity) and G is row-normalized, $\forall i \in \mathcal{N} : \sum_{j \in \mathcal{N}} G_{ij} = 1$ (assume no i is isolated).[44] This is the linear-in-means model. Here, the spectral radius equals one, $\mu = 1$. The Jacobian also becomes row diagonally dominant with $\delta_i = |\phi|$, and the

[41]The spectral radius of $|\phi G|$ is always weakly greater than of ϕG by Horn and Johnson (2012) theorem 8.1.18. Under pure strategic complements, $\forall i, j : \phi G_{ij} \geq 0$, the two spectral radii are equal because $\phi G = |\phi G|$.

[42]A network is strongly connected if any node in the network has a directed path to any other node (Bramoullé et al., 2016, page 523). This assumption is very common in the network literature, especially when considering learning dynamics, existence of a consensus, and eigenvector centralities (Bramoullé et al., 2016, section 19.3.2).

[43]Proposition 4 uses the matrix A from equation (67), which in the present model is $A_{ij} = |\phi G_{ij}|$, with $\rho = |\phi|\,\mu$. Note that $|G|$ is irreducible iff G is irreducible.

[44]It's straightforward to include isolated individuals. The iDD degree for an isolated individual is zero. The bounds in equations (137) and (138) are still valid but no longer sharp. The sharp bounds for isolated i and non-isolated j are $\frac{\partial y_i}{\partial x_i} = \beta$, $\frac{\partial y_i}{\partial x_j} = 0$, $\left|\frac{\partial y_j}{\partial x_i}\right| \leq |\phi|\left|\frac{\partial y_i}{\partial x_i} + \frac{\gamma}{\phi}\right|$.

bounds from proposition 3 can be applied to $\frac{\partial \tilde{y}_i}{\partial x_i}$, noting that $\frac{\partial f_i}{\partial x_i}$ satisfies equation (59)

$$\forall i \in \mathcal{N}: \quad \frac{\partial y_i}{\partial x_i} \in \left[\frac{\beta+\frac{\gamma}{\phi}}{1+\phi^2}, \frac{\beta+\frac{\gamma}{\phi}}{1-\phi^2}\right] - \frac{\gamma}{\phi} \tag{137}$$

$$\forall i \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\frac{\partial y_j}{\partial x_i}\right| \leq |\phi| \left|\frac{\partial y_i}{\partial a_i} + \frac{\gamma}{\phi}\right| \tag{138}$$

where again the case $\beta + \frac{\gamma}{\phi} \geq 0$ is shown (and again the bracketed term in equation 137 is flipped when $\beta + \frac{\gamma}{\phi} < 0$). If we further assume $\phi \geq 0$ (strategic complements), then the lower bound in equation (137) can be strengthened by noting that ∇ satisfies equation (73), one of the conditions for proposition 5. This implies that ∇ also satisfies equation (77), and therefore $\forall i : \{\nabla^{-1}\}_{ii} \geq 1$, thus[45]

$$\frac{\partial y_i}{\partial x_i} = \left\{\nabla^{-1}\right\}_{ii} \left(\beta + \frac{\gamma}{\phi}\right) - \frac{\gamma}{\phi} \begin{cases} \geq \beta & \text{if } \beta + \frac{\gamma}{\phi} \geq 0 \\ \leq \beta & \text{if } \beta + \frac{\gamma}{\phi} \leq 0 \end{cases}$$

where the first equality follows from equation (134). If one assumes $\operatorname{sgn}(\beta) = \operatorname{sgn}(\gamma)$ (direct and contextual effects have the same sign), then the comparative static for $j = i$ satisfies

$$\forall i \in \mathcal{N}: \quad \left|\frac{\partial y_i}{\partial x_i}\right| \in \left[|\beta|, \frac{|\beta| + \phi |\gamma|}{1-\phi^2}\right], \qquad \operatorname{sgn}\left(\frac{\partial y_i}{\partial a_i}\right) = \operatorname{sgn}(\beta) \tag{139}$$

where the upper bound used $\left|\frac{\partial y_i}{\partial x_i}\right| \leq \left|\frac{\beta+\frac{\gamma}{\phi}}{1-\phi^2} - \frac{\gamma}{\phi}\right| = \frac{|\beta|+\phi|\gamma|}{1-\phi^2}$, with the inequality following from applying equation (137) under both cases $\beta + \frac{\gamma}{\phi} \gtreqless 0$. Equation (24) in the main text is derived by setting $\gamma = 0$ in equation (139).

If one also assumes $|\gamma| \leq |\beta|$ (contextual effects are weaker than direct effects), which is likely the typical prior as the effect of a shock tends to decay with distance, equation (139) can be inverted to bound ϕ in terms of the comparative statics, $\frac{\partial y_i}{\partial x_i}$. Just like in the main text in section 4.1, use the upper bound for a subset individuals

[45] Proposition 5 cannot be applied completely because ∇ does not satisfy (column) diagonal dominance assumption 1), but instead row diagonal dominance equation (58). One, however, can derive an analogue to proposition 5 that applies to a row diagonally dominant ∇ that obeys the sign equation (73).

$\mathcal{N}_1 \subset \mathcal{N}$, which gives

$$i \in \mathcal{N}_1: \quad \left|\frac{\partial y_i}{\partial x_i}\right| \leq \frac{|\beta| + \phi\,|\gamma|}{1-\phi^2} \overset{|\gamma|\leq|\beta|}{\leq} |\beta|\,\frac{1+\phi}{1-\phi^2} = |\beta|\,\frac{1}{1-\phi}$$

And substitute β out using the lower bound in equation (139) for a different subset of individuals $\mathcal{N}_2 \subset \mathcal{N}$

$$i \in \mathcal{N}_1, j \in \mathcal{N}_2: \quad \phi \geq 1 - \left|\frac{\partial y_j}{\partial x_j} \Big/ \frac{\partial y_i}{\partial x_i}\right|$$

Which can be identified using

$$\phi \geq 1 - \left|\frac{b_2}{b_1}\right| \tag{140}$$

where b_1, b_2 are estimated from regression equation (27) as in the main text. The difference between the bound here and in the main text is that equation (26) requires the stronger assumption $\gamma = 0$, while equation (140) requires the weaker assumption $|\gamma| \leq |\beta|$. Accordingly, the lower bound under $\gamma = 0$ is weakly greater than under $|\gamma| \leq |\beta|$, $\sqrt{1 - \left|\frac{b_2}{b_1}\right|} \geq 1 - \left|\frac{b_2}{b_1}\right|$. Using the estimate of $\left|b_2/b_1\right| = 0.53$ from table 2, the lower bound under $|\gamma| \leq |\beta|$ is 0.47, as opposed to 0.69 under $\gamma = 0$. Intuitively, the lower bound on ϕ is less when allowing for contextual effects because some of the observed effect on y_i from x_i, $\frac{\partial y_i}{\partial x_i}$, may be driven by the contextual effects γ, and not by the endogenous peer effect ϕ.

C.6 Time Series

Consider the ARMA(p, q) model, a widely used econometric framework for time series (Brockwell and Davis, 2016). The nodes in this framework correspond to time periods, and interactions between nodes are described by the autoregressive lag coefficients. For the comparative statics of the dependent variable in the econometric model, I show assumption 1 is satisfied if the sum (in absolute terms) of all the autoregressive coefficients is less than one. This assumption guarantees existence and uniqueness of a stationary and causal solution. The bounds on the comparative static (theorem 1) with respect to the j^{th} lag of the independent variable in the econometric model depends only on the absolute sum of all the autoregressive coefficients, and the j^{th} lag coefficient.

Let $i \in \mathcal{N}$ represents time over N periods, and $\{y_i\}_{i\in\mathcal{N}}$ be a ARMA(p, q) process defined by the model

$$i \in \mathcal{N}: \quad y_i - \sum_{s=1}^{p} \beta_s y_{i-s} = x_i + \sum_{s=0}^{q} \theta_s x_{i-s} \tag{141}$$

where $\{x_i\}_{i\in\mathcal{N}}$ is a white noise process (definition 3.1.1 in Brockwell and Davis (2016)). β_s is the effect of y with a lag of s periods, and θ_s the effect of x with a lag of s periods. The equations of state in this model corresponds to

$$i \in \mathcal{N}: \quad f_i(\boldsymbol{y}, \boldsymbol{x}) = y_i - \sum_{s=1}^{p} \beta_s y_{i-s} - x_i - \sum_{s=1}^{q} \theta_s x_{i-s}$$

The Jacobian is

$$\nabla_{ij} = I_{ij} - \beta_{i-j} \cdot 1[i-p \le j < i] \tag{142}$$

The direct effects matrix is $\frac{\partial f_i}{\partial x_j} = -I_{ij} - \theta_{i-j} \cdot 1[i-q \le j < i]$, with

$$\left|\frac{\partial f_{-i}}{\partial x_j}\right| = \begin{cases} \sum_{s=1}^{q} |\theta_s| & j = i \\ 1 + \sum_{s \in \{1,\dots,q\}\setminus i} |\theta_s| & j \ne i \end{cases}$$

The Jacobian ∇ satisfies assumption 1 iff

$$\sum_{s=1}^{p} |\beta_s| < 1 \tag{143}$$

Proof. Assumption 1 is satisfied iif

$$|\nabla_{jj}| - \sum_{i \ne j} |\nabla_{ij}| > 0 \tag{144}$$

Under the Jacobian of this model,

$$\begin{aligned} |\nabla_{jj}| - \sum_{i \ne j} |\nabla_{ij}| &= 1 - \sum_{i \ne j} |I_{ij} - \beta_{i-j} \cdot 1[i-p \le j < i]| \\ &= 1 - \sum_{s=1}^{p} |\beta_s| \end{aligned}$$

where the first line used equation (142) and the second line a manipulation of the indices. Thus, equation (144) holds iff $\sum_{s=1}^{p} |\beta_s| < 1$. $\square$

The condition in equation (143) is sufficient for existence and uniqueness of a causal, stationary solution $\{y_i\}_{i\in\mathcal{N}}$ for this model as $N \to \infty$.

Proof. By Brockwell and Davis (2016) pg 75, a stationary solution exists and is unique iff

$$1 - \sum_{s=1}^{p} \beta_s z^s \neq 0 \tag{145}$$

for all $|z| = 1$ where $z \in \mathbb{C}$. The stationary solution is causal (y_i only depends on x_j for $j \leq i$) if equation (145) holds for all $|z| \leq 1$ where $z \in \mathbb{C}$ (Brockwell and Davis, 2016 pg 75). A sufficient condition for equation (145) is

$$Re \left\{ \sum_{s=1}^{p} \beta_s z^s \right\} \neq 1$$

This holds under equation (143) because

$$\begin{aligned} Re \left\{ \sum_{s=1}^{p} \beta_s z^s \right\} &\leq \left| \sum_{s=1}^{p} \beta_s z^s \right| \\ &\leq \sum_{s=1}^{p} |\beta_s| \, |z^s| \\ &\leq \sum_{s=1}^{p} |\beta_s| \\ &< 1 \end{aligned}$$

where the second line used the triangle inequality, the third used $|z| \leq 1$ and the last line used equation (143) . Thus (143) is sufficient for the existence and uniqueness of a causal stationary solution. $\square$

In what follows I will consider only nodes $i < N - \max\{p, q\}$ to avoid having to complicate the notation to include boundary cases. However, the results naturally apply to these nodes too.

The iDD degree (definition 1) is

$$\delta_i = \sum_{s=1}^{p} |\beta_s|$$

Under equation (143), theorem 1 can be used to imply bounds on the comparative static of y_i with respect to x_j. Suppose $\sum_{s=1}^{q} |\theta_s| \leq 1$, then

$$\frac{\partial y_i}{\partial x_j} \in \begin{cases} \left[\frac{1-\sum_{s=1}^{q}|\theta_s|\sum_{s=1}^{p}|\beta_s|}{1+\left(\sum_{s=1}^{p}|\beta_s|\right)^2}, \frac{1+\sum_{s=1}^{q}|\theta_s|\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}\right] & j = i \\ \left[\frac{\theta_{i-j}-\left(1+\sum_{s\in\{1,\ldots,q\}\setminus i}|\theta_s|\right)\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}, \frac{\theta_{i-j}+\left(1+\sum_{s\in\{1,\ldots,q\}\setminus i}|\theta_s|\right)\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}\right] & j \neq i \end{cases}$$

where $j \in [i-q, i]$. Note that the sign of $\frac{\partial y_i}{\partial x_j}$ is positive for $j = i$ while it's undetermined for $j \neq i$. The bounds are identified from knowledge of only the sums of all the coefficients on y, $\sum_{s=1}^{p} |\beta_s|$, and the x coefficient corresponding to the shock θ_{i-j}, and the sum of the remainder, $\sum_{s\in\{1,\ldots,q\}\setminus i} |\theta_s|$. Knowledge of all the coefficients individually is not required.

C.7 Spatial Econometrics

Consider the standard spatial autoregressive model, a workhorse framework in spatial econometrics (LeSage et al., 2009, section 3.1.1). The nodes in this framework correspond to locations, and the interactions between nodes are described by the spatial weight matrix and the spatial lag parameter. For the comparative statics of the dependent variable in the econometric model, I show row diagonal dominance (remark 3) is satisfied if the spatial lag parameter is less than one in absolute value. This condition is assumed almost without exception in the literature. The bounds on the comparative static with respect to the independent variable in any location is identified from the spatial lag parameter and the coefficient on the exogenous local characteristic alone; notably, the spatial weight matrix is not needed to calculate the bounds. I show how this can be used to partially identify the spatial lag parameter under incomplete or misspecification of the spatial weight matrix, which helps remedy a key critique against the spatial econometric literature (Gibbons and Overman, 2012).

The spatial autogressive model is

$$y_i = \rho \sum_{j \in \mathcal{N}} W_{ij} y_j + \beta x_i + \varepsilon_i \tag{146}$$

where $i \in \mathcal{N}$ represents locations, y and x are some endogenous and exogenous location characteristics, respectively. $W_{ij} \geq 0$ is the spatial weight matrix, representing how strongly connected locations i and j are, often parameterized to be inversely related to distance between the two locations. $\rho \in \mathbb{R}$ is the spatial lag (or spatial autoregressive) parameter, modulating the overall magnitude and sign of the spatial interdependence. β is the direct effect of x_i on y_i, holding the indirect effect through $\sum_{j \in \mathcal{N}} W_{ij} y_j$ fixed. ε_i is the residual.

Typically in these models, the spatial weight matrix is constructed to have no interaction within a location, $\forall i \in \mathcal{N} : W_{ii} = 0$, and all row sums equal to one, $\forall i \in \mathcal{N} : \sum_{j \in \mathcal{N}} W_{ij} = 1$. The term $\sum_{j \in \mathcal{N}} W_{ij} y_j$ then represents a spatially weighted average of y neighboring i. Under this restriction, the parameter ρ is almost always assumed to have modulus less than one (Ord, 1975; Kelejian and Robinson, 1995; LeSage et al., 2009; Gibbons and Overman, 2012)

$$|\rho| < 1 \tag{147}$$

This assumption implies the model has a "stable" solution (Fingleton, 1999). That is, the variance of the system is finite even if the matrix has cycles or if $N \to \infty$. It also ensures the model is well-defined as $I - \rho W$ is invertible. The prevalence of this assumption, however, is perhaps more rooted in computational convenience, facilitating maximum likelihood estimation. The equations of state in this model are

$$0 = f_i(\boldsymbol{y}, \boldsymbol{x}) \equiv y_i - \rho \sum_{j \in \mathcal{N}} W_{ij} y_j - \beta x_i - \varepsilon_i$$

The Jacobian is

$$\nabla_{ij} \equiv I_{ij} - \rho W_{ij}$$

and the direct effects matrix is $\frac{\partial f_i}{\partial x_j} \equiv -\beta I_{ij}, \left|\frac{\partial f_{-i}}{\partial x_j}\right| = \beta (1 - I_{ij})$. Equation (147), along with the fact that all row sums of (nonnegative) W equal one, implies that the

Jacobian is row diagonally dominant,

$$\forall i \in \mathcal{N}: \quad \sum_{j \notin \mathcal{N}} |\nabla_{ij}| = \sum_{j \notin \mathcal{N}} |\rho| W_{ij} < \sum_{j \notin \mathcal{N}} W_{ij} = 1 = |\nabla_{ii}|$$

And because the only non-zero direct effect is on $j = i$, equation (59) is satisfied and we can apply proposition 3 to bound the comparative static. The iDD degree under row diagonal dominance is

$$\delta_i = \frac{\sum_{j \notin \mathcal{N}} |\nabla_{ij}|}{|\nabla_{ii}|} = \frac{\sum_{j \notin \mathcal{N}} |\rho| W_{ij}}{1} = |\rho|$$

The resulting bounds are

$$\frac{\partial y_i}{\partial x_i} \in \left[\frac{\beta}{1+\rho^2}, \frac{\beta}{1-\rho^2}\right] \tag{148}$$

and

$$j \neq i: \quad \left|\frac{\partial y_j}{\partial x_i}\right| \leq |\rho| \left|\frac{\partial y_i}{\partial x_i}\right| \leq \frac{|\rho| \beta}{1-\rho^2} \tag{149}$$

Where the sign is determined for the $j = i$ comparative static because equation (10) is satisfied. Note that the second inequality in equation (149) used the upper bound of equation (148).

The bounds are enumerated from knowledge of β, ρ only; no knowledge of the spatial weight matrix, W, is required. This is a potentially useful insight, as a difficulty in spatial autoregressive models is characterizing the implications of different ρ for the distribution of y, which is generally challenging due its interaction with the spatial weight matrix, W (Conley, 2016 pg 7).

A key critique of the spatial econometric literature is that identification of ρ depends pivotally of the specification of the spatial weight matrix W being correct (e.g. the assumed functional form of its dependence on distance), which is typically hard to justify (Gibbons and Overman, 2012). My bounds can be used to partially identify ρ even if W is misspecified, thus alleviating this critique.

Restricting $\rho \in [0, 1)$, one can apply the same strategy as in step 9 of section 4.1, as the spatial econometric model equation (146) is isomorphic to the linear-in-means model equation (14). One would use the regression in equation (27) for two subsets of locations that differ in their position in the spatial weight matrix. For instance, a measure correlated with centrality in W (such as trade openness or market access, depending on the empirical setting) would suffice. The result is a lower bound on ρ (analogue to equation 26) that does not require correct or full specification of W.