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TRADING AMBIGUITY: A TALE OF TWO HETEROGENEITIES*

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We consider markets with heterogeneously ambiguous assets and heterogeneously ambiguity-averse investors whose preferences are a parsimonious extension of the mean–variance framework. We study portfolio choice and trade upon arrival of public information, and show systematic departures from the predictions of standard theory, that occur in the direction of empirical regularities. In particular, our theory speaks to several phenomena in a unified fashion: the asset allocation puzzle, the observation that earnings announcements are followed by significant trading volume with small price change, and that increases in uncertainty are positively associated with increased trading activity and portfolio rebalancing toward safer assets.

1. INTRODUCTION

Modern decision theory uses the term *ambiguity* to describe uncertainty about a data-generating process. The decision maker believes that the data come from an unknown member of a set of possible models. Knight (1921) and Ellsberg (1961) intuitively argue that concern about this uncertainty induces a decision maker to want decision rules that work robustly across the set of models believed to be possible. The argument is formalized in pioneering contributions by Schmeidler (1989) and Gilboa and Schmeidler (1989) followed by a body of subsequent work including robust control theory (Hansen and Sargent, 2008) and the theory of smooth ambiguity aversion (Klibanoff et al., 2005).

The financial literature largely proceeds from the assumption that investors behave as if they know the distributions of returns, ruling out ambiguity. However, this assumption is hard to justify. Finer sampling would, arguably, virtually eliminate estimation errors for second moments of return distributions, but it is well established that first moments (i.e., means) are extremely difficult to estimate (Merton, 1980; Blanchard, 1993; Cochrane, 1997; Anderson et al., 2003). This article considers investors who are concerned about the ambiguity of return distributions, more specifically, the ambiguity due to the uncertainty about the means of returns.[1] We conceive of this *parameter uncertainty* in a Bayesian fashion: unknown means are treated as random variables. More concretely, combining a prior over the means for the set of assets

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[1] Trautmann and van de Kuilen (2015) surveys experimental evidence on ambiguity aversion. Dimmock et al. (2016) and Bianchi and Tallon (2019) use survey and experimental data to relate evidence on individuals' ambiguity aversion to their behavior in financial markets, for instance, portfolio holdings. Chew et al. (2018) document that ambiguity aversion is more prevalent among analytically sophisticated subjects.

considered with observations from the data, we take the resulting posterior joint distribution of the means to describe the parameter uncertainty, that is, the ambiguity about the return distributions.[2] The ambiguity-averse investor is inclined to choose a portfolio position whose value is less affected by, and hence robust to, the parameter uncertainty.

We use *smooth ambiguity*, in particular, a specification that simplifies to *robust mean–variance preferences* (Maccheroni et al., 2013), to model investors' ambiguity aversion.[3] If the data-generating process were known, then a smooth ambiguity-averse investor would evaluate a portfolio by its expected utility given that process. However, given the parameter uncertainty, the posterior joint distribution of the asset mean returns together with the portfolio induces a distribution over possible expected utility values. If one takes two portfolio positions a and a' such that the distribution of expected utility values induced by a' is a mean-preserving spread of the distribution induced by a, then the smooth ambiguity-averse investor prefers a to a'. The (smooth) ambiguity-neutral investor, on the other hand, is indifferent between two such portfolios and hence maximizes his expected utility by investing in a Bayesian optimal portfolio.[4]

Recent contributions have demonstrated ambiguity aversion can significantly help to explain empirical regularities involving price of aggregate uncertainty.[5] In these papers, effectively, there is a single (ambiguously) uncertain asset and homogeneous ambiguity aversion. Our article investigates market implications when there are *multiple uncertain* assets with heterogeneous ambiguity, that is, assets that can be ranked in terms of uncertainty about mean returns. The article considers, in a unified setting, portfolio choice and trade upon arrival of public information, given such heterogeneity in the cross section of assets. We find that there are significant departures from the predictions of standard theory, given the presence of another key ingredient—a second heterogeneity: multiple agents who are heterogeneously ambiguity averse. Importantly, these departures occur in the direction of empirical regularities that belie the standard theory.

Our theory (supported by quantitative exercises calibrated to data) explains and connects two significant sets of evidence. The first is a set of pervasive and puzzling observations about investor behavior and financial market outcomes which in and of themselves make no reference to ambiguity aversion. One such observation is the so-called asset allocation puzzle, which refers to the finding that in contrast to the mutual fund theorem, the very common advice from financial advisors is that aggressive investors should hold a lower ratio of bonds to stocks than conservative investors in their risky portfolios (Canner et al., 1997). Another observation is that earnings announcements are often followed by significant trading volume with small price change (Kandel and Pearson, 1995). Also, Giannetti and Laeven (2016) observe that investors rebalance their portfolios toward geographically close firms during periods of high market volatility and that firms with higher local ownership exhibit less sensitivity to innovations in market-wide implied volatility (as measured by changes in VIX). That is, investors change the composition of their *uncertain* portfolios in a particular way at times of high market volatility. In a similar vein, Kostopoulos et al. (2021) show that increases in ambiguity, measured by positive innovations in the 30-day implied volatility of VSTOXX,[6] lead to increased trading activity and individual (retail) investors reducing their exposure to the security market by trading out of stocks and similarly risky assets.

[2] An alternative conception of parameter uncertainty, in the classical instead of Bayesian approach, is offered by Garlappi et al. (2007) in the form of *confidence intervals* around the point estimates of means.

[3] See Cubitt et al. (2020), and references to the literature therein, for experimental evidence comparing the smooth ambiguity model to other alternatives.

[4] The pioneering works of Klein and Bawa (1976) and Brown (1979) study portfolio choice and pricing implications of parameter uncertainty by modeling investors as choosing Bayesian optimal portfolios.

[5] For instance, see Maenhout (2004), Epstein and Schneider (2008), Illeditsch (2011), Condie et al. (2020), Mele and Sangiorgi (2015) and in a more macro-finance tradition, Ju and Miao (2012), Hansen and Sargent (2010), and Collard et al. (2018).

[6] VSTOXX is the European equivalent of VIX and based on the Euro Stoxx index. Kostopoulos et al. (2021) also show that their results are robust to alternative survey-, newspaper-, or market-based ambiguity measures.

There is also a second set of evidence from surveys and field experiments that relates ambiguity aversion to household portfolio choices: Dimmock et al. (2016) show that higher ambiguity aversion is associated with lower portfolio fractions allocated to equity and also find that investors with higher ambiguity aversion were significantly more likely to actively sell equities during the 2008–09 financial crisis. Bianchi and Tallon (2019) match administrative panel data on portfolio choices with survey data on preferences over ambiguity and produce similar results to Dimmock et al. (2016).

Our specification of the smooth ambiguity model simplifies to a parsimonious extension of the standard mean–variance framework—it adds to this standard formula a third term, involving ambiguity, so that the investor faces a three-way trade-off between expected return, risk, and ambiguity. As intuition suggests, we demonstrate through comparative statics exercises that more ambiguity-averse investors resolve this trade-off by putting more weight on ambiguity and thus are more partial to the less ambiguous assets. This key, single driving force connects our results on portfolio choice to equilibrium asset prices and trade upon arrival of public information.

In the theory developed here, the nature of trading is dictated by ambiguity-sharing considerations. Following the public signal, the return–risk–ambiguity trade-off changes, making investors seek a different allocation of more and less ambiguous assets depending on their different tolerances for ambiguity. This requires portfolio rebalancing, that is, changes in the composition of the portfolio of uncertain assets, thus mutually beneficial exchange of such assets. In particular, larger uncertainty shocks cause individual portfolios to move further away from the market portfolio, leading to larger trading volumes.

As we noted, investors' response to heterogeneity in the ambiguity of assets is key to our results. Intuitively, this heterogeneity may arise for at least a couple of reasons. One reason may be that some assets are structurally more exposed to uncertainty quite generally, whether it be risk or ambiguity. For instance, a firm's stock return is structurally more exposed to uncertainty than its bond return as stock is a residual claim. Moreover, bond returns are exposed to only downside uncertainty whereas stock returns are exposed to both downside and upside uncertainty. A second reason is about the fundamentals of the underlying asset. For instance, new-technology companies or companies exploring new markets would have fundamentals whose risks have not been fully learned. Also, firms which are more exposed to aggregate uncertainty shocks, for instance, because of financial distress or reliance on external financing, would be in this category. The success of the macrofinance literature incorporating ambiguity in explaining price dynamics lends support to the idea of treating aggregate uncertainty as ambiguous.[7] In that literature, the assumed source of the ambiguity in the agent's beliefs is the occurrence of periodic, temporary changes in the probability distribution governing next period's growth outcome due to the effect of the business cycle. More precisely, according to this view, macroeconomic (i.e., systematic) ambiguity is essentially the evolving uncertainty about where the economy is with respect to the business cycle; that is, uncertainty about how big the temporary departure of the mean growth rate from the trend growth rate is and how long this departure will last.

The channel governing portfolio choice and trade we study implies a persistence of a pattern of heterogeneity in portfolio allocations among more and less ambiguity-averse investors, so long as ambiguity persists. And, following episodes of heightened uncertainty, the heterogeneity in holdings gets exacerbated. Recent empirical evidence has documented striking patterns of persistence in households' investment choices, with potentially serious implications for persistence of wealth inequalities. As noted by Buss et al. (2021), "one group of households tilts its investments toward safe and familiar assets, trend chases, and earns lower investment returns, while another group of households holds riskier positions and exhibits superior market-timing abilities, consequently earning higher investment returns" (Bianchi, 2018,

[7] See Gallant et al. (2019), in particular, for an assessment of the models applying the smooth ambiguity preference framework to this context.

Fagereng et al., 2020). These differences in investment behavior persist for remarkably long periods and are a crucial determinant of the dynamics of wealth inequality (Piketty, 2014, Bach et al., 2020, Fagereng et al., 2020). In addition, they have substantial welfare implications for individuals and also for society (Campbell, 2016, Bhamra and Uppal, 2019). These welfare implications provide a reason for an improved understanding of the channel we study in the article.

We close this introduction with a brief description of our contribution in terms of theoretical model-building. We follow Hara and Honda (2022) in setting up our static model: the specification of the assets, and agents' beliefs and preferences. Hara and Honda (2022) note that the mutual fund theorem fails when there is heterogeneity in ambiguity aversion. We go beyond them, first, by introducing a second heterogeneity—-the Jewitt and Mukerji (2017) notions of one asset being more ambiguous than another—and applying it to characterize the departure from the mutual fund theorem. Second, we build on the insights developed in this characterization to investigate how the interaction between the two heterogeneities, those of ambiguity aversion and asset ambiguity, affects equilibrium trade following public announcements. To that end, we extend the model of Hara and Honda (2022), which is static, to a dynamic setup incorporating two different types of public signals by applying the recursive smooth ambiguity framework of Klibanoff et al. (2009).

The article is organized as follows: In Section 2, we describe the setup we adopt and describe investors' preferences and common beliefs about asset returns. Then in Section 3, we analyze portfolio choice in a static setting with two uncertain assets and a risk-free asset. In Section 4, we propose two dynamic extensions of the static model in order to study how prices and trade respond to the arrival of public information. In the first dynamic extension, in the interim period the agents receive a public signal drawn from the same process which governs the realization of the final-period return. We interpret this signal as an earnings announcement. In the second dynamic extension, we consider uncertainty shocks: the public signal is an event which directly increases or decreases the parameter uncertainty (i.e., ambiguity). Section 5 elaborates on the connection between the mechanism we explore in this article and that of subjectivity of beliefs in an expected utility framework in the contexts of portfolio choice and trade following public signals. Section 6 concludes. Appendix A.1 contains details on the calibration of the quantitative exercises. Proofs of the results and lemmas can be found in Appendix A.2.

2. THE BASE SETUP

In this section, we describe the domain of choice, agents, their beliefs, and preferences. We consider a model with two uncertain assets $i = 1, 2$ and a risk-free asset f. The price of the risk-free asset, p_f, is normalized to 1. The price of uncertain asset i is denoted by p_i. There is a finite set of agents, $\{1, \ldots, n, \ldots, N\}$. Agent n's holdings of the assets are denoted by $q_{i,n}$; for convenience, we will write $q_n = (q_{1,n}, q_{2,n})$.[8] We denote by $a_{i,n} \equiv p_i q_{i,n}$ the monetary amount invested in asset i by agent n, and write $a_n = (a_{1,n}, a_{2,n})$ to denote the monetary investment (or equivalently, monetary holdings) in uncertain assets. Given the normalization $p_f = 1$, $a_{f,n} = q_{f,n}$ denotes the monetary holding of the risk-free asset by agent n. Agent n's endowment of asset i is $e_{i,n}$, and the aggregate endowment of the asset is e_i. All agents have zero endowment of the risk-free asset, and therefore there is zero aggregate supply of the risk-free asset so that $e_f = 0$. Both risk-free and uncertain returns are exogenous. The gross (monetary) returns of the risk-free asset and uncertain assets are R_f and R_i, $i = 1, 2$, respectively. We let $R \equiv (R_1, R_2)$. If agent n invests $a_{j,n}$ in asset j, where $j = f, 1, 2$, then his payoff is $R_j a_{j,n}$.

The uncertain returns are ambiguous in the sense that agents are uncertain about the probability distribution governing each return: they believe that the returns data are generated by

[8] All vectors are taken to be column vectors. Transposes are row vectors.

an unknown member of a set of possible models. Formally, we have a random vector $M \equiv (M_1, M_2)$, the model, whose realization fixes the vector of conditional distribution of returns $R|M \equiv (R_1|M_1, R_2|M_2)$. The uncertainty about returns conditional on a model M (i.e., $R|M$) is referred to as the *first-order uncertainty* whereas uncertainty about the model M itself is referred to as the *second-order uncertainty*. We adapt the setup of Hara and Honda (2022) to describe the agents' common beliefs about the uncertainty governing returns.[9] In this setup, both first- and second-order uncertainties are Gaussian. We further impose the following assumptions:[10]

Assumption 1. *The mean return of asset i conditional on model M is M_i, $i = 1, 2$. That is,*

$$\mathrm{E}[R \mid M] = M.$$

Assumption 2. *Models and asset returns are jointly normally distributed with*

$$cov(R, M) = var(M) \equiv \Sigma_M.$$

That is,

$$\begin{pmatrix} M \\ R \end{pmatrix} \sim N\left(\begin{pmatrix} \mathrm{E}[M] \\ \mathrm{E}[R] \end{pmatrix}, \begin{pmatrix} \Sigma_M & \Sigma_M \\ \Sigma_M & \Sigma_R \end{pmatrix}\right),$$

where

$$\Sigma_M = \begin{pmatrix} (\sigma_1^M)^2 & \sigma_{12}^M \\ \sigma_{12}^M & (\sigma_2^M)^2 \end{pmatrix} \text{ and } \Sigma_R = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & {\sigma_2}^2 \end{pmatrix}.$$

Assumption 2 is actually all about the joint normality of R and M; the restriction that $\mathrm{cov}(R, M) = \mathrm{var}(M)$ does not result in loss of generality as pointed out by Hara and Honda (2022). The matrix Σ_R is the variance–covariance matrix of the unconditional distribution of returns R. Henceforth, we will refer to $(\sigma_i^M)^2$ and σ_i^M as the *model variance of asset i* and *model standard deviation of asset i*, respectively, and Σ_M as the model variance–covariance matrix. The Projection Theorem, together with Assumptions 1 and 2, yields

$$M = \mathrm{E}[R \mid M] = \mathrm{E}[R] + [\mathrm{cov}(R, M)][\mathrm{var}(M)]^{-1}(M - \mathrm{E}[M]) = \mathrm{E}[R] + M - \mathrm{E}[M],$$

which in turn implies that $\mathrm{E}[R] = \mathrm{E}[M] \equiv \mu \equiv (\mu_1, \mu_2)$. We assume that $\mu_i > R_f$, $i = 1, 2$, so that risk- and ambiguity-averse agents do not rule out investment into uncertain assets. Also, following Assumption 2 and the Projection Theorem, we have

$$\Sigma \equiv \mathrm{var}(R \mid M) = \mathrm{var}(R) - [\mathrm{cov}(R, M)][\mathrm{var}(M)]^{-1}[\mathrm{cov}(R, M)] = \Sigma_R - \Sigma_M. \tag{1}$$

Hence, we obtain that the conditional asset return is distributed as

$$R \mid M \sim N(M, \Sigma_R - \Sigma_M).$$

[9] Assuming common (second-order) beliefs has the consequence, as shown by Rigotti et al. (2008), that ambiguity-averse agents will not want to enter into speculative trades, in the sense that, absent aggregate uncertainty, the full insurance allocations are Pareto optimal.

[10] We slightly abuse notation by denoting both the model random variable and a particular realization of the variable by M.

In particular, the variance of returns conditioned on the realization of M is independent of the realized value. In this setup, the uncertainty about the model *only* affects the (conditional) mean of the return, not the (conditional) variance, and thus reduces to *parameter uncertainty* about the mean.

We apply the framework of the smooth model of Klibanoff et al. (2005) to describe how the agents incorporate uncertainty in the evaluation of portfolios. In this model, if the agent were to know the realization of M and so faced no parameter uncertainty, the evaluation is the usual expected utility evaluation using the random variable $R|M$. Uncertainty about M makes this expected utility evaluation (based on $R|M$) uncertain; the agent is ambiguity averse if he dislikes mean-preserving spreads in this uncertainty. More specifically, consider a portfolio $(a_{f,n}, a_{1,n}, a_{2,n})$ which yields a final contingent wealth equal to $W(a_{f,n}, a_{1,n}, a_{2,n}) = a_{f,n}R_f + a_{1,n}R_1 + a_{2,n}R_2$. Agent n evaluates such a portfolio according to:[11]

$$\mathrm{E}_M[\phi_n(\mathrm{E}_{R|M}[u_n(W(a_{f,n}, a_{1,n}, a_{2,n}))])], \tag{2}$$

where u_n, a utility function, incorporates the agent's attitude to risk, and ϕ_n, an increasing concave function, reflects the agent's ambiguity aversion. Thus, the ambiguity and the ambiguity aversion are represented distinctly through the random variable M and the function ϕ_n, respectively. This parametric separation is useful in that it is possible to hold an agent's beliefs (perceived ambiguity) fixed while varying their ambiguity attitude, say from aversion to neutrality (i.e., replacing a concave ϕ_n with an affine one reduces the preference to expected utility while retaining the same beliefs).[12] As in Hara and Honda (2022), we further specify $u_n(x) = -\exp(-\theta_n x)$ and $\phi_n(y) = -(-y)^{\gamma_n/\theta_n}$. Denote by

$$\eta_n \equiv -y\,\frac{\phi_n''(y)}{\phi_n'(y)} = \frac{\gamma_n}{\theta_n} - 1 = \frac{\gamma_n - \theta_n}{\theta_n},$$

the coefficient of (relative) ambiguity aversion of the agent. Note, if $\gamma_n = \theta_n$, then agent n is ambiguity neutral and a CARA (expected) utility maximizer. If $\gamma_n > \theta_n$, the agent is ambiguity averse. In the rest of the article, we will always assume that $\eta_n \geq 0$ for all n, that is, we never consider ambiguity seeking.[13]

We now set up the maximization problem the agents solve. Given Assumptions 1 and 2, and the specifications of u_n and ϕ_n as above, Lemma 1 of Hara and Honda (2022) shows that maximizing (2) is equivalent to choosing a portfolio $(a_{f,n}, a_{1,n}, a_{2,n})$ that maximizes

$$V_n(a_{f,n}, a_{1,n}, a_{2,n}) \equiv \underbrace{a_{f,n}R_f + \mu^\top a_n}_{\text{mean return}} - \frac{\theta_n}{2}\underbrace{a_n^\top \Sigma_R a_n}_{\text{risk}} - \frac{\gamma_n - \theta_n}{2}\underbrace{a_n^\top \Sigma_M a_n}_{\text{ambiguity}}, \tag{3}$$

where $a_n = (a_{1,n}, a_{2,n})$. This formulation therefore generalizes the standard and commonly used mean–variance model of Markowitz (1952): $\frac{\theta_n}{2}\, a_n^\top \Sigma_R a_n$ is the standard risk adjustment to the evaluation whereas $\frac{\gamma_n - \theta_n}{2}\, a_n^\top \Sigma_M a_n$ introduces an ambiguity adjustment. Maccheroni et al. (2013) obtain the formulation (3), which they refer to as the *robust mean–variance model*, as

[11] E_X denotes an expectation operator which integrates over the realization of the random variable X.

[12] An ambiguity neutral, standard expected utility agent cares only about the unconditional uncertainty, represented by the random variable R.

[13] Consider two agents in our model, $n = 1, 2$, who share the same beliefs about returns and have attitudes parameterized by the tuple (η_n, θ_n). Since $\eta_n = \frac{\gamma_n}{\theta_n} - 1$, only two of the three parameters, η_n, γ_n, and θ_n, may be independently varied when comparing preferences (see also Klibanoff et al., 2005, pp. 1867–68). So, the two agents have the same ambiguity aversion but agent 1 is more risk averse than agent 2 if $\eta_1 = \eta_2$ but $\theta_1 > \theta_2$. Note that this necessarily implies $\gamma_1 = (\eta_1 + 1)\,\theta_1 > (\eta_2 + 1)\,\theta_2 = \gamma_2$. It is worth emphasizing that η_n represents ambiguity aversion but γ_n does not. For instance, consider the case where $\gamma_1 > \gamma_2$, $\gamma_1 < \theta_1$, and $\gamma_2 > \theta_2$. Even though $\gamma_1 > \gamma_2$, agent 1 is ambiguity seeking ($\eta_1 < 0$) whereas agent 2 is ambiguity averse ($\eta_2 > 0$).

a second-order (Arrow–Pratt) approximation of the certainty equivalent of a smooth ambiguity evaluation where u_n, ϕ_n and beliefs are arbitrarily specified. This is analogous to the fact that the standard mean–variance formulation may be seen as a quadratic approximation of the certainty equivalent of an expected utility evaluation where utility and beliefs are arbitrarily specified.

Observe, we may rewrite the formulation in (3) as

$$V_n(a_{f,n}, a_{1,n}, a_{2,n}) = a_{f,n}R_f + \mu^\top a_n - \frac{\theta_n}{2} a_n^\top (\Sigma_R + \eta_n \Sigma_M) a_n. \tag{4}$$

Hence, our ambiguity-averse agent's evaluation of the portfolio can be read as if it were the evaluation of a *standard* (as opposed to *robust*) mean–variance utility agent with absolute risk-aversion parameter θ_n and an *as if* belief that uncertain assets' return distribution is given by $N(\mu, \Sigma_R + \eta_n \Sigma_M)$.[14] Thus, our population of robust mean–variance agents with identical beliefs but heterogeneous ambiguity aversion may be equivalently seen as a population of standard mean–variance agents with heterogeneous *as if* beliefs which differ in the variance *but not in the mean* of returns. The disagreement in as if beliefs is in the term $\eta_n \Sigma_M$, hence it stems from differences in η_n and the differences are magnified by Σ_M. Given that the ambiguity is about the mean returns, one might have expected that more ambiguity-averse decision makers would behave as if they believe the mean returns were lower, but instead they behave as if the return variances are higher. This as if beliefs interpretation is helpful in gaining intuition for some of the results we obtain in the subsequent analysis.[15]

3. PORTFOLIO CHOICE AND THE ASSET ALLOCATION PUZZLE

3.1. *Portfolio Choice.* We study here how the composition of the optimal portfolio is determined by ambiguity aversion given exogenous asset prices. Consider agent n with initial wealth $W_n \equiv p_1 e_{1,n} + p_2 e_{2,n}$. The maximization problem he faces is

$$\max_{a_{f,n},\ a_{1,n},\ a_{2,n}} \quad a_{f,n}R_f + \mu^\top a_n - \frac{\theta_n}{2} a_n^\top \Sigma_R a_n - \frac{\gamma_n - \theta_n}{2} a_n^\top \Sigma_M a_n \tag{5}$$
$$\text{s.t.} \quad a_{f,n} + a_{1,n} + a_{2,n} \le W_n.$$

Solving for the optimal monetary holdings of uncertain assets, $a_n = (a_{1,n}, a_{2,n})$, yields:

$$a_n = \frac{1}{\theta_n}(\Sigma_R + \eta_n \Sigma_M)^{-1}(\mu - R_f 1), \tag{6}$$

or more explicitly,

$$a_{i,n} = \frac{(\mu_i - R_f)A_{j,n} - (\mu_j - R_f)B_{12,n}}{A_{1,n}A_{2,n} + (B_{12,n})^2}, \quad i = 1, 2, \tag{7}$$

[14] This does not mean that the overall behavior of our ambiguity-averse investor can be mimicked by a subjective expected utility maximizer. Consider, for instance, the following two pairs of derivative assets, each pair describing a complementary bet on the return of an underlying asset: in particular, one derivative's payoff per dollar invested is 1 if the return of the underlying asset is greater than its mean (according to the unconditional distribution) and 0 otherwise whereas the other derivative has opposite payoffs. Let A, A^c be the derivatives corresponding to the unambiguous underlying asset whereas let B, B^c be those corresponding to the ambiguous underlying asset whose unconditional mean return is equal to that of the unambiguous asset. Then our investor will exhibit the preference $A \succ B$ and $A^c \succ B^c$ which is inconsistent with subjective expected utility.

[15] That the mechanism of ambiguity aversion may act through the channel of as if beliefs has been observed by, for instance, Hansen and Sargent (2008, p. 9, para. 3), Strzalecki and Werner (2011), Gollier (2011), and Collard et al. (2018, Remark 1) in relation to robust control theory, uncertainty sharing, portfolio choice, and asset pricing, respectively.

where

$$A_{i,n} = \theta_n \sigma_i^2 + (\gamma_n - \theta_n)(\sigma_i^M)^2 \; = \; \theta_n\big[\sigma_i^2 + \eta_n(\sigma_i^M)^2\big],$$
$$B_{12,n} = \theta_n \sigma_{12} + (\gamma_n - \theta_n)\sigma_{12}^M \; = \; \theta_n\big[\sigma_{12} + \eta_n \sigma_{12}^M\big].$$

Therefore, we have

$$\frac{a_{1,n}}{a_{2,n}} = \frac{(\mu_1 - R_f)\big[\sigma_2^2 + \eta_n(\sigma_2^M)^2\big] - (\mu_2 - R_f)\big[\sigma_{12} + \eta_n\sigma_{12}^M\big]}{(\mu_2 - R_f)\big[\sigma_1^2 + \eta_n(\sigma_1^M)^2\big] - (\mu_1 - R_f)\big[\sigma_{12} + \eta_n\sigma_{12}^M\big]}. \tag{8}$$

So, the ratio of monetary investments in uncertain assets is independent of the agent's risk-aversion parameter θ_n. If everyone is ambiguity neutral (i.e., $\eta_n = 0$ for all n), this implies nothing but the classical mutual fund theorem (Tobin, 1958): ambiguity-neutral agents, regardless of their risk aversion, invest in the same proportion across uncertain assets, and therefore hold the same portfolio of uncertain assets. The ratio in (8) does depend, however, on the agent's ambiguity-aversion parameter η_n.[16] The mutual fund theorem continues to hold if agents are homogeneously ambiguity averse. However, generically, two agents n and n' with different ambiguity-aversion parameters will have different ratios of monetary investments in uncertain assets. Noting $\frac{a_{1,n}}{a_{2,n}} = \frac{p_1 q_{1,n}}{p_2 q_{2,n}}$, this also means that they will hold uncertain assets in different proportions, that is, $\frac{q_{1,n}}{q_{2,n}} \neq \frac{q_{1,n'}}{q_{2,n'}}$. The following remark summarizes these observations:

REMARK 1. If agents are homogeneous in ambiguity aversion, that is, $\eta_n = \eta_{n'}$ for all n, n', then the mutual fund theorem holds, that is, for optimal portfolio choices it holds that $\frac{a_{1,n}}{a_{2,n}} = \frac{a_{1,n'}}{a_{2,n'}}$ and $\frac{q_{1,n}}{q_{2,n}} = \frac{q_{1,n'}}{q_{2,n'}}$ for all n, n'. If, on the other hand, agents are heterogeneous in ambiguity aversion, then the mutual fund theorem generically fails.

When agents are homogeneously ambiguity averse, Equation (4) tells us they act as if they have standard mean–variance preferences with the same beliefs, and hence the mutual fund theorem follows. On the other hand, agents with heterogeneous ambiguity aversion act as if they have different beliefs about return variances, which leads them to hold different portfolios as they disagree on how to optimally diversify. The latter point is made explicit by (6): optimal monetary holdings are a function of $\eta_n \Sigma_M$, the distinguishing aspect of the as if belief.

That the mutual fund theorem holds with homogeneous ambiguity aversion and fails with heterogeneous ambiguity aversion was already noted in Hara and Honda (2022) and can be inferred from Ruffino (2014). In the next subsection, we go beyond this observation by characterizing the departure from the mutual fund theorem in terms of agents' ambiguity aversion and ambiguity of the assets.

3.2. *Comparative Statics.* Let $S_i \equiv \frac{\mu_i - R_f}{\sigma_i}$ and $S_i^{Amb} \equiv \frac{\mu_i - R_f}{\sigma_i^M}$, $i = 1, 2$. S_i is, of course, the standard Sharpe ratio of asset i, and we will refer to S_i^{Amb} as the *ambiguity Sharpe ratio* of asset i. We also let $\rho \equiv \frac{\sigma_{12}}{\sigma_1\sigma_2}$ and $\rho^M \equiv \frac{\sigma_{12}^M}{\sigma_1^M\sigma_2^M}$. Lemma A.1 in the Appendix gives a full characterization of the comparative statics of portfolio choice with respect to ambiguity aversion. The characterizing condition in Lemma A.1 is a mouthful, but at its heart lies a three-way trade-off between excess return, risk, and ambiguity—glimpsed through the interplay between the (standard and ambiguity) Sharpe ratios and the two correlation terms ρ and ρ^M. We explore the content of the characterization through Proposition 1 and a couple of corollaries, to follow.

[16] Unless nongeneric restrictions on the asset return structure hold, such as, obviously, if the two assets are identical with $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\sigma_1^M = \sigma_2^M$, or if $\sigma_i = \sigma_i^M$ and $\sigma_{12} = \sigma_{12}^M$.

PROPOSITION 1. *Let agent n be more ambiguity averse than agent n′, that is,* $\eta_n > \eta_{n'}$. *Then, for optimal portfolio choices it holds that*

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \textit{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}},$$

if either of the following conditions holds:

(a) $S_1 = S_2$, $S_1^{Amb} < S_2^{Amb}$, *and* $\rho \neq 1$, *or*
(b) $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$ *and* $\rho(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}) \leq \rho^M \leq \rho(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2})$.

In (a), we make the agent indifferent between the two assets in terms of return compensation per unit risk, but partial to asset 2 in terms of return compensation per unit ambiguity. If there is no room for risk diversification (i.e., if $\rho = 1$), then absent ambiguity aversion, agent's choice between the uncertain assets would be indeterminate, hence we could not say how more ambiguity aversion would affect the agent's choice. However, when $\rho \neq 1$, risk diversification uniquely determines the optimal portfolio choice absent ambiguity aversion, and introducing ambiguity aversion tilts the optimal choice away from asset 1. Therefore, the agent would hold proportionately less of asset 1 if he were more ambiguity averse.

In (b), the restriction $\rho(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}) \leq \rho^M \leq \rho(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2})$ implies that the correlations ρ and ρ^M have the same sign, and thus, that diversification opportunities in risk are aligned, as would seem plausible, with those in ambiguity. Assuming such an alignment ensures that the trade-off between risk diversification and ambiguity diversification does not play a significant role in heterogeneously ambiguity-averse agents' portfolio choices. The first-order effect that leads to differences in differently ambiguity-averse agents' portfolios, instead, comes from these agents' different evaluations of each asset's ambiguity in comparison to its risk-return profile. Specifically, if asset 1 offers lower return per unit ambiguity in proportion to return per unit risk compared to asset 2, that is, if $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$, then the more ambiguity-averse agent holds proportionately less of asset 1.

Intuition suggests that more ambiguity-averse agents should put more weight on ambiguity in the three-way trade-off between return, risk, and ambiguity and therefore be more partial to the less ambiguous assets. We articulate this intuition more precisely in the following corollaries, obtained by applying two formal notions of one asset being more affected by ambiguity than another, developed by Jewitt and Mukerji (2017). Given the class of preferences in our setup, asset 1 is *more ambiguous (I)* than asset 2 if and only if $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\sigma_1^M > \sigma_2^M$, and asset 1 is *more ambiguous (II)* than asset 2 if and only if $\sigma_1^M > \sigma_2^M$.[17] Using this new vernacular, we have the following result as a direct consequence of Proposition 1:

COROLLARY 1. *Let agent n be more ambiguity averse than agent n′, that is,* $\eta_n > \eta_{n'}$. *If*

(i) *asset 1 is more ambiguous (I) than asset 2 and* $\rho \neq 1$, *or*
(ii) *asset 1 is sufficiently more ambiguous (II) than asset 2 so that* $\sigma_1^M > \sigma_2^M(\frac{\sigma_1}{\sigma_2})$ *and* $\rho(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}) \leq \rho^M \leq \rho(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2})$,

then for optimal portfolio choices it holds that

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \textit{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}.$$

[17] For characterizations of both "more ambiguous (I)" and "more ambiguous (II)" for the class of preferences considered here, see Example 4.2 in Jewitt and Mukerji (2017).

TABLE 1
ASSET ALLOCATIONS RECOMMENDED BY FINANCIAL ADVISORS

	Percent of Portfolio			
Advisor and Investor Type	Cash	Bonds	Stocks	Ratio of Bonds to Stocks
A. Fidelity				
Conservative	50	30	20	1.50
Moderate	20	40	40	1.00
Aggressive	5	30	65	0.46
B. Merrill Lynch				
Conservative	20	35	45	0.78
Moderate	5	40	55	0.73
Aggressive	5	20	75	0.27
C. Jane Bryant Quinn				
Conservative	50	30	20	1.50
Moderate	10	40	50	0.80
Aggressive	0	0	100	0.00
D. New York Times				
Conservative	20	40	40	1.00
Moderate	10	30	60	0.50
Aggressive	0	20	80	0.25

Next, we take one uncertain asset to be more ambiguous (II) and to have a lower ambiguity Sharpe ratio compared to the other one, and ask how an agent would optimally allocate her wealth between the uncertain assets if she were *sufficiently* ambiguity averse.

COROLLARY 2. *Let $S_1^{Amb} < S_2^{Amb}$. If agent n is sufficiently ambiguity averse, and asset 1 is more ambiguous (II) than asset 2, then the agent optimally allocates a smaller portion of his wealth to asset 1 compared to asset 2, that is, $a_{1,n} < a_{2,n}$.*

3.3. *Asset Allocation Puzzle.* Following Corollaries 1 and 2, the departure from the mutual fund theorem is in a particular direction in that more ambiguity aversion leads to a portfolio that has proportionately more of the less ambiguous asset. This seems to accord well with the widely recognized deviation from the mutual fund theorem, the *asset allocation puzzle*, first noted in Canner et al. (1997): it is very common to observe in financial planning advice that more conservative investors are encouraged to hold more bonds, relative to stocks. Table 1 reproduced from Canner et al. (1997) illustrates the puzzle.

As documented by Canner et al. (1997) and as expected, stocks are riskier than long-term government bonds which are in turn riskier than Treasury bills, where risk is proxied by standard deviation. It follows that the standard error of the estimate of mean returns for each asset class is also ranked the same way. Hence, the confidence interval around the estimated mean returns gets wider as we move from Treasury bills to bonds to stocks, suggesting a greater degree of uncertainty or poorer knowledge about what the true mean is. This in turn suggests stock returns may be perceived as more ambiguous than bond returns and bond returns may be perceived as more ambiguous than Treasury bill returns. Within the standard mean–variance framework, the mutual fund theorem holds and therefore interpreting conservatism as risk aversion does not explain the asset allocation puzzle. On the other hand, the popular financial advice is accommodated in the robust mean–variance framework when we interpret conservatism as ambiguity aversion.

We now provide an exercise that numerically illustrates how an investor's optimal allocation ratio of bonds to stocks depends on his ambiguity aversion. Table II in Canner et al. (1997) reports distribution parameters of 1926–92 real annual returns for Treasury bills, bonds, and stocks, and the authors use these parameter values to numerically assess how relaxing underlying assumptions of the mutual fund theorem would affect optimal-portfolio allocations.

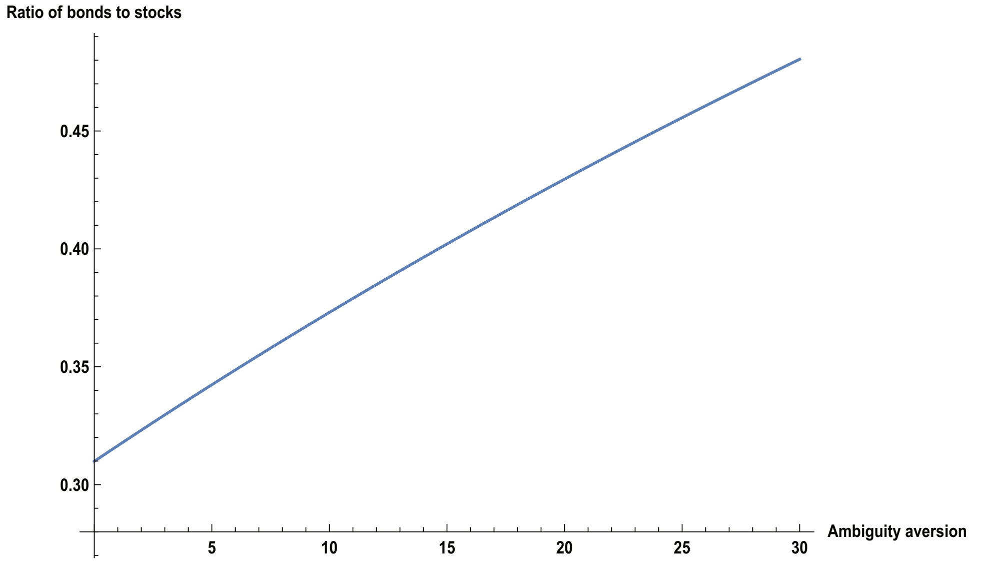

FIGURE 1

THE GRAPH PLOTS AN INVESTOR'S OPTIMAL ALLOCATION RATIO OF BONDS TO STOCKS IN RELATION TO THE INVESTOR'S AMBIGUITY-AVERSION COEFFICIENT

Using the parameter values given and implied by this table, we derive the optimal allocation ratio of bonds to stocks from (8) and plot the derived ratio in Figure 1 as a function of the ambiguity-aversion coefficient (see Appendix A.1 for details on the parameterization). In particular, the optimal allocation ratio is found to be approximately 0.31, 0.40, and 0.48 when ambiguity-aversion coefficient is set as 0, 15, and 30, respectively. These ratios are qualitatively similar and quantitatively comparable to the ratios reported in Table 1 if we interpret conservatism as ambiguity aversion.[18]

Canner et al. (1997) discuss various possible explanations of the asset allocation puzzle (though not ambiguity aversion) and find them unsatisfactory. In particular, they point out that (see pp. 183-184 of Canner et al. (1997)) subjective beliefs cannot be an explanation, as presumably the financial advisor's subjective belief about asset returns does not change depending on whom they advise.[19] In their concluding remarks, interestingly, the authors conjecture that nonstandard preferences may help explain the puzzle. Dimmock et al. (2016) test the relation between ambiguity aversion and the fraction of financial assets allocated to stocks, and find it to be statistically and economically significantly negative, lending support to our explanation. A distinctive feature of our explanation based on the interaction between heterogeneity in asset ambiguity and agent ambiguity aversion is that it also provides a basis for candidate explanations of the nature of trade following earnings announcements and uncertainty shocks, as we show in subsequent analysis.

[18] Collard et al. (2018), which applies the smooth ambiguity preference framework to explain aggregate price dynamics in a macrofinance context, provides a perspective regarding the plausibility of the range of ambiguity-aversion coefficients used in our numerical exercise, though ambiguity aversion is modeled as an exponential function instead of power as here: table 2 of the said paper shows that the first and second moments of the risk-free rate and the equity premium observed in the data are close to the theoretical predictions for these moments when a range of plausible risk-aversion coefficients (1–5) and a range of calibrated ambiguity-aversion coefficients (6.65–31.5) are employed.

[19] Typically, when financial advisors make their portfolio recommendations to their clients, they do so by gauging their clients' general tolerance for uncertainty, not their subjective beliefs about individual asset returns.

3.4. *Discussion: Robustness of Results.* In this section, we discuss the robustness of our comparative statics results, in particular Proposition 1, to our modeling assumptions.

Robustness to specifications of beliefs and preferences. First, as noted, Equation (3) is a robust mean–variance formulation. Hence, if an agent had a robust mean–variance preference, Proposition 1 obtains irrespective of the parametric form of agents' beliefs, in particular, agents' beliefs need not be Gaussian.

Second, when uncertainty is small in a particular sense, the comparative static results in Proposition 1 hold for smooth ambiguity preferences quite generally. Campbell (2018, subsection 2.1), using an Arrow–Pratt methodology considers a *risk with small reward* to derive an approximate formula for dollars invested in the risky asset in the standard portfolio problem with one safe and one risky asset and (arbitrary) expected utility preferences. It is then shown that this formula is exact for mean–variance preferences. We can show an analogous result that relates the ratio of amounts invested in the two uncertain assets in the portfolio choice problem considered in Proposition 1 with robust mean–variance preferences, to the optimal ratio of amounts invested in the same portfolio choice problem with more general smooth ambiguity preferences.

More precisely, let

$$\begin{aligned} R &\equiv (R_1, R_2) &= (R_f + X_1, R_f + X_2), \\ X &\equiv (X_1, X_2) &= (M_1 + \epsilon_1, M_2 + \epsilon_2), \\ M &\equiv (M_1, M_2) &= (km_1 + \zeta_1, km_2 + \zeta_2), \end{aligned}$$

and assume that $k, m_1, m_2 > 0$ are deterministic scalars and ϵ_i and ζ_i, $i = 1, 2$, are random variables with $\mathrm{E}[\epsilon_i] = \mathrm{E}[\zeta_i] = 0$. This rewriting of the parameters of the model is without loss of generality.

In this context, X_i is the excess return of asset i and M_i is the ambiguous mean of this excess return. Assume that u_n exhibits constant absolute risk aversion. Then, for any arbitrary distribution of model parameters, M and $R|M$, and an arbitrary smooth ambiguity-aversion function ϕ_n, Proposition 1 holds for sufficiently small k, that is, if uncertain assets yield sufficiently small reward for their uncertainty.[20]

Robustness to the number of assets. Proposition 1 provides a characterization of the comparative statics of portfolio choice with respect to ambiguity aversion and, to this end, uses Sharpe ratio and ambiguity Sharpe ratio of assets as characterizing conditions. However, if the number of uncertain assets is arbitrary, then the portfolio choice is made according to each asset's contributions to portfolio risk and portfolio ambiguity. As we know from modern portfolio theory (mean–variance analysis), the former contribution is measured by the asset's portfolio beta. To measure the latter, similar to the definition of portfolio beta, one can define *portfolio ambiguity beta* where the covariance and variance operators in the definition are applied to means of returns instead of returns. There is a one-to-one correspondence between the Sharpe ratios and portfolio betas when there are only two uncertain assets, hence for that case the contributions to portfolio risk and portfolio ambiguity are fully captured by the Sharpe ratios. Note that, with mean–variance preferences, all agents hold the market portfolio as their uncertain portfolio and therefore an asset's portfolio beta is the market (portfolio) beta and does not vary across agents. However, as we have shown, with heterogeneity in attitudes toward ambiguity, agents' uncertain portfolio compositions differ. As a consequence, in our model assets' portfolio betas and portfolio ambiguity betas differ across agents, depending on their ambiguity aversion. Hence, it is not possible to give conditions on betas analogous to those on Sharpe ratios as stated in Proposition 1 (as the latter is an asset-specific measure whereas the former depends on both the asset and the agent) and be able to extend the comparative statics to an arbitrary number of assets.

[20] Proof of the assertion is available from the authors upon request.

4. DYNAMIC EQUILIBRIUM ANALYSIS WITH PUBLIC SIGNALS

We now turn to two dynamic extensions of our static model where we study how prices and trade respond to the arrival of a *public* signal. The two dynamic extensions differ in the content of this signal. In the first extension, the agents receive a public signal drawn from the same process which governs the realization of the liquidating dividends of uncertain assets. Upon receiving the signal, the agents update their beliefs about the distribution of the model M. We interpret these signals as earnings announcements. In the second dynamic extension, the signal informs only about the model variance; we see the realization of such a signal as a realization of a (model) uncertainty shock. The next subsection presents the dynamic structure common to both extensions.

4.1. *The Common Dynamic Structure and Notion of Equilibrium.* We model a three-period economy whose structure and timeline are as follows:

- In the initial period, $t = 0$, agents trade and choose a portfolio of three assets, two uncertain (indexed by 1,2) and one risk-free (indexed by f).
- In the interim period, $t = 1$, agents receive a public signal $S = (S_1, S_2)$ about the liquidating dividends of the uncertain assets, update their beliefs, and have an opportunity to trade again in all the assets. The risk-free asset pays off R_f for each dollar invested in $t = 0$. No dividend from the uncertain assets is realized, however their prices change endogenously following the signal. No consumption takes place in this period.
- In the final period, $t = 2$, no decisions are taken—the risk-free asset pays off R_f for each dollar invested in $t = 1$, liquidating dividends of the uncertain assets realize (more precisely, asset i pays off R_i for each dollar invested in $t = 1$), and agents consume.

In $t = 0$, asset i's ($i = 1, 2, f$) price is denoted by p_i^0, its quantity held by agent n is $q_{i,n}^0$, and the corresponding monetary holding is $a_{i,n}^0 \equiv p_i^0 q_{i,n}^0$. In $t = 1$, these variables depend on the realization of the signal. When talking about price and holdings conditional on the realization of a signal S, we write p_i^S, $q_{i,n}^S$, and $a_{i,n}^S \equiv p_i^S q_{i,n}^S$. The price of the risk-free asset is normalized to 1 in $t = 0, 1$. Uncertain asset i's ($i = 1, 2$) gross return from $t = 0$ to $t = 1$ is equal to $\frac{p_i^S}{p_i^0}$. We refer to this as asset i's *interim return*. Abusing notation, we let $\frac{p^S}{p^0} \equiv [\frac{p_1^S}{p_1^0}, \frac{p_2^S}{p_2^0}]$. The uncertain asset i pays off R_i in $t = 2$ for each dollar invested in $t = 1$. We refer to this as asset i's *return*. Note that the asset i pays off $\frac{p_i^S}{p_i^0} R_i$ in $t = 2$ for each dollar invested in $t = 0$. As in the static model, the aggregate endowment of the uncertain asset i is denoted by e_i—to simplify, we take the endowment to be the same for both uncertain assets so that $e_i = e$ for $i = 1, 2$. There is zero aggregate supply of the risk-free asset so that $e_f = 0$.

An *equilibrium* is given by prices and holdings of uncertain and risk-free assets in periods 0 and 1 such that the holdings are optimal, given the prices and information, and clear the markets in both periods. We refer to equilibrium prices and holdings in period 0 and in period 1 as *ex ante equilibrium* and *interim equilibrium*, respectively. We say that an *equilibrium entails trivial trading* if the composition of the uncertain portfolio stays the same across periods 0 and 1, but the quantity of the risk-free asset held changes. An *equilibrium entails nontrivial trading* if the composition of the uncertain portfolio changes from period 0 to period 1. Formally,

- an *equilibrium entails trivial trading at signal realization S* if, at the equilibrium,

$$\frac{q_{1,n}^S}{q_{2,n}^S} = \frac{q_{1,n}^0}{q_{2,n}^0} \text{for all } n \text{ and } q_{f,n'}^S \neq q_{f,n'}^0 \text{ for some } n',$$

- an *equilibrium entails nontrivial trading at signal realization* S if, at the equilibrium,

$$\frac{q_{1,n}^S}{q_{2,n}^S} \neq \frac{q_{1,n}^0}{q_{2,n}^0} \text{for some } n,$$

- an *equilibrium entails no trade at signal realization* S if it entails neither trivial nor nontrivial trading.

We now define interim and ex ante preferences according to the recursive smooth ambiguity formulation of Klibanoff et al. (2009) which, given the recursive construction, guarantees dynamically consistent behavior. After realization of the signal S, the agent updates his beliefs through Bayes rule. Let $M' \equiv M|S$ and $R' \equiv R|S$ denote the updated beliefs over M and R. Then, the interim utility from an interim portfolio $(a_{f,n}^S, a_n^S)$ is given by

$$U_n^S(a_{f,n}^S, a_n^S) \equiv \phi_n^{-1}\Big(\mathrm{E}_{M'}\Big[\phi_n(\mathrm{E}_{R'|M'}\Big[u_n\Big(W_n^2(a_{f,n}^S, a_n^S)\Big)\Big])\Big]\Big), \tag{9}$$

where $W_n^2(a_{f,n}^S, a_n^S) = (a_n^S)^\top R + a_{f,n}^S R_f$ is the final wealth obtained after the liquidation of the dividends. Ex ante, prior to the realization of the signal, the utility from an initial portfolio $(a_{f,n}^0, a_n^0)$ is, via recursion as stipulated by Klibanoff et al. (2009),

$$U_n^0(a_{f,n}^0, a_n^0) \equiv \phi_n^{-1}\Big(\mathrm{E}_M\Big[\phi_n\Big(\mathrm{E}_{S|M}\Big(U_n^S(a_{f,n}^{*,S}, a_n^{*,S})\Big)\Big)\Big]\Big), \tag{10}$$

where $(a_{f,n}^{*,S}, a_n^{*,S})$ is a solution to

$$\max_{a_n^S, a_{f,n}^S} U_n^S(a_{f,n}^S, a_n^S) \tag{11}$$

subject to the budget constraint

$$(a_n^S)^\top 1 + a_{f,n}^S \leq (a_n^0)^\top \frac{p^S}{p^0} + a_{f,n}^0 R_f \equiv W_n^S(a_{f,n}^0, a_n^0). \tag{12}$$

Observe that $(a_{f,n}^{*,S}, a_n^{*,S})$ depends on $(a_{f,n}^0, a_n^0)$. In (9) and in (10), we assume $u_n(x) = -\exp(-\theta_n x)$ and $\phi_n(y) = -(-y)^{\gamma_n/\theta_n}$ as in the static analysis of Section 2. Note, (9) is a robust mean–variance formulation whereas (10) is a *recursive* robust mean–variance formulation.

4.2. *Earnings Announcements.*

4.2.1. *Modeling earnings announcements.* We formalize an earnings announcement as a publicly observed signal drawn from the same stochastic process governing uncertain asset returns. This signal allows the agents to update their common prior on models believed to generate returns—thus leaving them better informed about returns. We let $S = (S_1, S_2)$ be the public signal about uncertain assets 1 and 2. Conditional on a model M, return R, and signal S are i.i.d. Consistent with the notation in (1) of the static setup, we let $\Sigma = \Sigma_R - \Sigma_M$ where Σ_R and Σ_M are as given in Assumption 2, and assume that the beliefs conditional on the model are

$$\begin{pmatrix} R|M \\ S|M \end{pmatrix} \sim N\left(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}\right). \tag{13}$$

As is the case in the static setup, let $M \sim N(\mu, \Sigma_M)$. Then, it follows from Bayes' Rule that $M' \equiv M|S \sim N(\mu_S, \Sigma_S)$, where

$$\mu_S = \left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\left(\Sigma_M^{-1}\mu + \Sigma^{-1}S\right), \tag{14}$$

$$\Sigma_S^{-1} = \Sigma_M^{-1} + \Sigma^{-1}. \tag{15}$$

Notice that μ_S is a linear function of S whereas Σ_S does not depend on the realized value of S. However, the precision of model uncertainty increases following the signal as evident from (15), where the left-hand side shows the precision of M' whereas the precision of M is given by Σ_M^{-1}. Analogous to Assumption 2 of the static setup, we assume that $\mathrm{cov}(R', M') = \mathrm{var}(M')$. Hence, $\mathrm{var}(R') \equiv \Sigma_{R'} = \Sigma + \Sigma_S$ and the updated beliefs are given by

$$\begin{pmatrix} M' \\ R' \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_S \\ \mu_S \end{pmatrix}, \begin{pmatrix} \Sigma_S & \Sigma_S \\ \Sigma_S & \Sigma + \Sigma_S \end{pmatrix}\right). \tag{16}$$

Also, following from (16), $R'|M' \sim N(M', \Sigma)$, which is analogous to (1) in the static model.

4.2.2. *Equilibrium analysis.* We solve the equilibrium backward, first deriving the interim equilibrium and then the ex ante. For the interim analysis, we place ourselves at period 1 once S is realized and observed by all agents. Recalling the interim maximand (11), the equivalent robust mean–variance form (3), the updated beliefs (16), and the budget constraint (12), the maximization problem of agent n reduces to

$$\max_{a_n^S}\left\{\left(W_n^S(a_{f,n}^0, a_n^0) - (a_n^S)1\right)R_f + (a_n^S)^\top\mu_S - \frac{\theta_n}{2}(a_n^S)^\top(\Sigma + \Sigma_S)a_n^S - \frac{\gamma_n - \theta_n}{2}(a_n^S)^\top\Sigma_S a_n^S\right\}, \tag{17}$$

where $W_n^S(a_{f,n}^0, a_n^0) = (a_n^0)^\top \frac{p^S}{p^0} + a_{f,n}^0 R_f$ is the wealth agent n derives from his portfolio $(a_{f,n}^0, a_n^0)$ when S is realized.

The interim equilibrium is characterized in Lemmas A.2 and A.3. The main takeaways from these results are as follows: agents hold the market portfolio in the interim period if they are homogeneous in ambiguity aversion. Otherwise, they generically hold different uncertain portfolios which vary with the realized signal S and their ambiguity aversions. Also, interim prices are linear functions of the signal.

Ex ante, agent n seeks to maximize $U_n^0(a_{f,n}^0, a_n^0)$ as defined in (10)—and where $(a_{f,n}^{*,S}, a_n^{*,S})$ is a solution to (17)—subject to the budget constraint $(q_n^0)^\top p^0 + q_{f,n}^0 \le W_n^0$, in which W_n^0 is agent n's wealth at time 0. Lemmas A.4 and A.5 characterize the ex ante equilibrium. We find that, under homogeneous ambiguity aversion, agents hold the market portfolio in the initial period and their risk-free holdings remain the same across both initial and interim periods. On the other hand, if agents are heterogeneous in ambiguity aversion, ex ante they hold uncertain portfolios which vary only with their ambiguity aversions. Recall that interim portfolios not only depend on ambiguity aversion but also vary with the signal realization. Therefore, there is no trade with homogeneity in ambiguity aversion whereas equilibrium generically entails nontrivial trading with heterogeneity—as we formally state in the following proposition:

PROPOSITION 2. *If agents are homogeneous in ambiguity aversion, that is, $\eta_n = \eta_{n'}$ for all n, n', then the equilibrium entails no trade at any signal realization. The equilibrium entails nontrivial trading at almost all signal realizations if agents are heterogeneous in ambiguity aversion, that is, if there exist n, n' such that $\eta_n \neq \eta_{n'}$.*

Recall from (4), heterogeneously ambiguity-averse agents can be interpreted as mean–variance agents with different as if beliefs, different only with respect to the variance term. On the other hand, the economy with homogeneously ambiguity-averse agents can be formally interpreted as a standard mean–variance economy with common beliefs, and is therefore *effectively complete* with everyone holding the market portfolio before and after the public signal.[21] This implies that the ex ante equilibrium allocation is Pareto-efficient, hence following a public signal there is no trade (not even trivial trading) under homogeneous ambiguity aversion.

For an intuition of why nontrivial trading arises with heterogeneous ambiguity aversion, recall that interim portfolio choice problem is the same as the static one, adapted to updated beliefs. As formalized in (16), upon the realization of a signal S, the beliefs get updated and therefore the ratio of interim period monetary holdings, $\frac{a^S_{1,n}}{a^S_{2,n}}$, varies with S.[22] And, a key takeaway from our static analysis is that this ratio would also vary with ambiguity aversion given any signal realization S. The ratio of initial-period monetary holdings, $\frac{a^0_{1,n}}{a^0_{2,n}}$, on the other hand, does not depend on the signal. Hence, the interim and initial ratios differ for almost all signal realizations. Can the intertemporal change in the asset prices fully account for the intertemporal change in the ratio of monetary holdings? The answer is negative if agents are heterogeneously ambiguity averse so that $\eta_n \neq \eta_{n'}$ for some n, n': then, for a given signal realization, the intertemporal change in the ratio of asset prices which would completely account for the intertemporal change in the ratio of monetary holding of agent n cannot be the same as the change in the ratio of prices that would fully account for the ratio change in the monetary holding of agent n'.

Differences in as if beliefs, under heterogeneous ambiguity aversion, explains why there is trade, but the idea is less helpful in understanding the *nontrivial* trade posited in Proposition 2.[23] For that, an ambiguity-sharing perspective takes us further. Following the public signal, the return–risk–ambiguity trade-off changes, making agents seek a different allocation of more and less ambiguous assets depending on their different tolerances for ambiguity. This prompts portfolio rebalancing and thus trade in uncertain assets. Moreover, in the ambiguity-sharing perspective the trade is mutually beneficial, that is, welfare increasing.

4.2.3. *Trade with and without price movements.* Kandel and Pearson (1995) document that earnings announcements are usually followed by a significant rise in trading volume—not necessarily associated with large price changes:

> *Using the announcement dates of quarterly (interim) earnings from the Compustat quarterly files and daily data on the returns and volumes of common stocks, we find that there are economically and statistically significant positive abnormal volumes associated with quarterly earnings announcements even when prices do not change in response to the announcements. It is notable that there appear to be abnormal volumes that are unrelated to the magnitudes of the price changes. This is inconsistent with most existing models of volume around public announcements in which agents have identical interpretations of public signals.*

By *abnormal volume*, the authors refer to the additional volume due to the public information release beyond the usual volume observed in no-announcement days which results from trading due to life-cycle considerations or trading to exploit private information. Since there is

[21] See the discussion of Rubinstein (1974) in Back (2017, pp. 54–58).

[22] While adapting the static portfolio choice characterization in (8) to the interim signal, the vector of parameters $(\mu_1, \mu_2; \sigma_1, \sigma_2, \sigma_{12}; \sigma_1^M, \sigma_2^M, \sigma_{12}^M)$ gets updated and the update varies with S.

[23] Caskey (2009) briefly refers to the effect of ambiguity on the possibility of trade following release of public information (see Proposition 2(c)): the decrease in ambiguity brought about by the public signal can generate (what we call, trivial) trade between the investors despite their having concordant beliefs. A more recent paper, Condie et al. (2020), shows that ambiguity-averse agents' optimal portfolios do not always depend on public information that is worse than expected and hence the equilibrium asset price does not reflect such information. What crucially drives this result is the ambiguity of the correlation between the asset payoff and the public signal. Such ambiguity is not modeled in our article (as we focus on the ambiguity of the means) and therefore neither optimal portfolios nor equilibrium prices underreact to worse-than-expected public signals in our model.

no life-cycle consideration or private information in our model, the trading volume attained in our analysis is "abnormal" in this sense.

We saw our theory gives an explanation for nontrivial trade following public signals (earnings announcements). But what does it have to say about price changes associated with such trade? To that end, first we have the following proposition:

PROPOSITION 3. *With heterogeneous ambiguity aversion, the equilibrium generically entails nontrivial trading at the signal realization which yields no price change across periods.*

Since equilibrium prices and holdings are continuous in the signal, it follows from the proposition that, given heterogeneous ambiguity aversion, in the neighborhood of the signal which yields no price change, there are equilibria entailing nontrivial trading with small price changes. To get an idea of the quantitative significance of the trading volume and associated price changes, later on in this subsection we report a numerical exercise. Before we do so, we discuss how our theory of trade following public announcements stands in relation to the existing literature.

A number of papers explain trading volume in dynamic settings with heterogeneous prior beliefs.[24] The mechanism in our article and the insights developed are distinct from those in the heterogeneous beliefs literature. In this literature, the article closest to ours in terms of results is Cao and Ou-Yang (2009) as it obtains nontrivial trade with little or no price change following a public signal. What crucially drives the result in Cao and Ou-Yang (2009) is that the agents' beliefs about asset payoffs *following* the realization of public signal differ *both* in means and variances, not just variances. Note, in our model the equivalent as if subjective beliefs following the signal differ only in the variances—therefore, the mechanism driving trade is different. The trading result in Cao and Ou-Yang (2009) is driven by speculation, that is, disagreement about posterior (conditional on the signal) mean payoffs. On the other hand, the mechanism we posit is driven by diversification or ambiguity-sharing needs. The ambiguity-sharing approach makes it absolutely clear that the trade is Pareto-improving, because in this approach the agents have common beliefs.[25]

In a setting with heterogeneous beliefs, speculative trades (i.e., side-bets) are arguably *ex ante* Pareto-improving—for example, see Christensen and Qin (2014). However, this idea has been contested by Mongin (1995, 2016) and Gilboa et al. (2014). In our setting, there is no possible controversy since beliefs are common. Thus, the trading mechanism driven by ambiguity sharing and the one driven by speculation generate two distinct insights regarding welfare implications; in turn, these have differing policy implications about the desirability of regulation to curb trade around earnings announcements.

We now provide the numerical exercise that reports data, as shown in Figure 2, on interim returns and abnormal trading volume generated by simulated signals in a calibrated dynamic economic equilibrium (see Appendix A.1 for details on the parameterization used to simulate signals). The graph in the left panel of Figure 2 shows substantial abnormal trading vol-

[24] In Harris and Raviv (1993), trade cannot occur in the absence of a price change. However, in Kandel and Pearson (1995) and Banerjee and Kremer (2010) such trade can occur if and only if agents have different priors about the mean of the public signal, thus trade is driven by agents' speculatively betting against each other. Also, in the latter two papers the economies consist of one uncertain asset and one risk-free asset, and therefore any trade generated in these papers is what we refer to as trivial trade in our context. Another strand of literature (e.g., Kim and Verrachia, 1994; He and Wang, 1995) explains trading volume in dynamic settings with heterogeneous information. In these models, there is no trading volume due to public announcements unless agents also have private information. Also, importantly there is no trade without an associated change in price contrary to the key empirical observation of Kandel and Pearson (1995). Ai and Bansal (2018) has a comprehensive analysis of the effects of public announcements on the equity premium with general recursive preferences including preferences under ambiguity. It is a representative agent model, with no analysis of trade.

[25] Werner (2022) shows that speculative trade may result from ambiguous but homogeneous beliefs when agents have heterogeneous hedging needs. As the article points out, the latter is "a critical condition for generating disagreement of effective beliefs with a common set of priors."

NOTES: The graph in the right panel shows the frequency distribution of (interim period) returns for asset 1.

FIGURE 2

THE GRAPH IN THE LEFT PANEL PLOTS ABNORMAL TRADING VOLUME IN RELATION TO (INTERIM PERIOD) ABSOLUTE RETURNS FOR ASSET 1 FOR A THOUSAND DRAWS OF PUBLIC SIGNALS

ume which is relatively unvarying—staying within a narrow band of the 2–3% of the aggregate endowment and accompanied by both low and high (interim period) absolute returns. Prices may spike or dive following public signals, but more often than not the price changes are modest (see the graph in the right panel of Figure 2). Hence, echoing the observation of Kandel and Pearson (1995), the more frequent occurrence is relatively small interim period returns accompanied by 2–3% abnormal trading volume.

What accounts for these findings? Recall from Subsection 4.2.2 that interim period prices are linear functions of public signal realizations, hence the distribution of interim returns follows that of signals. Extreme interim returns are low probability events, occurring only following signal realizations which are surprises. Small interim returns are more frequent and follow signal realizations which are closer to agents' expectations. The trading volume, on the other hand, follows the portfolio adjustments prompted by the change in return–risk–ambiguity trade-off following the signals. The change in ambiguity does not depend on the realized value of the signal, unlike interim returns. Thus, even when interim returns are small, the more common occurrence, the trading volume can still be significant. In summary, the price change is smaller the more the signal confirms expectations, whereas the trading volume is mostly dictated by the change in ambiguity.

4.3. *Uncertainty Shocks.* We next explore the issue of reaction to arrival of information in a somewhat more stylized model, where the signal realizations directly report the ambient ambiguity. More precisely, the signal directly determines the variance of second-order beliefs (i.e., model uncertainty) but not the means of the returns. This modeling strategy allows us to look into the relation between changes in the level of ambiguity and trading volume unencumbered by any change in mean returns. Therefore, the analytical staging of this extension is different from that in the previous one where mean returns and the level of ambiguity were affected by the signal but only the mean returns varied with the realized value of the signal.

To fix ideas, one may think of such signals as uncertainty shocks: shocks which determine the level of uncertainty in the environment. As a concrete example, think of the Brexit vote outcome as one of two possible signal realizations: Brexit or the status quo. Each realization could determine a distinct level of uncertainty; in particular, a larger parameter uncertainty would follow the Brexit outcome.

4.3.1. *Modeling uncertainty shocks.* We consider a version of the model introduced in Subsection 4.1 with a special signal structure to model uncertainty shocks—there are three possible realizations of the signal, which we call interim states, $S = H(\text{igh}), I(\text{ntermediate}), L(\text{ow})$, that directly inform on the level of ambiguity: the defining feature of an interim state is the associated model variance denoted by $\bar{\Sigma}_S$. We assume, for simplicity, that the signal realizations are unambiguous events, that is, the probability of S, denoted by $\pi(S)$, is the same under any model M. Like in the previous extension, we let $R|M \sim N(M, \Sigma)$ where $\Sigma = \Sigma_R - \Sigma_M$.

As before, we denote by $M' = M|S$ and $R' = R|S$ the updated beliefs over M and R and assume that $\text{cov}(R', M') = \text{var}(M')$. Hence, as was the case in (16), we have

$$\begin{pmatrix} M' \\ R' \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \bar{\Sigma}_S & \bar{\Sigma}_S \\ \bar{\Sigma}_S & \Sigma + \bar{\Sigma}_S \end{pmatrix}\right)$$

and $R'|M' \sim N(\mu, \Sigma)$. Note, however, unlike in (16), $E[M']$ and $E[R']$ do not depend on S.

4.3.2. *Equilibrium analysis and implications.* We assume only two agents, $n = 1, 2$, in our equilibrium analysis. As in Subsection 4.2.1, we start with the interim period maximization program of agent n, once the signal is realized:

$$\max_{a_n^S} \left\{ \left(W_n^S(a_{f,n}^0, a_n^0) - (a_n^S)1 \right) R_f + (a_n^S)^\top \mu - \frac{\theta_n}{2}(a_n^S)^\top (\Sigma + \bar{\Sigma}_S) a_n^S - \frac{\gamma_n - \theta_n}{2}(a_n^S)^\top \bar{\Sigma}_S a_n^S \right\},$$

where $W_n^S(a_{f,n}^0, a_n^0) = (a_n^0)^\top \frac{p^S}{p^0} + a_{f,n}^0 R_f$ is the wealth agent n derives from his portfolio $(a_{f,n}^0, a_n^0)$ when S is realized. The solution of this maximization problem for uncertain assets is

$$a_n^{*,S} = \frac{1}{\theta_n}(\Sigma + \bar{\Sigma}_S + \eta_n \bar{\Sigma}_S)^{-1}(\mu - R_f 1). \tag{18}$$

The interim period market clearing condition is $\sum_n a_{i,n}^S = p^S \sum_n e_{i,n}$. Given our assumption that asset endowments are equal, $\sum_n e_{1,n} = \sum_n e_{2,n} \equiv e$, we have that at an interim equilibrium

$$p^S = \frac{1}{e} \sum_n \frac{1}{\theta_n}(\Sigma + \bar{\Sigma}_S + \eta_n \bar{\Sigma}_S)^{-1}(\mu - R_f 1) \tag{19}$$

and asset holding in interim period state S is

$$q_{i,n}^{*,S} = \frac{a_{i,n}^{*,S}}{p_i^S}. \tag{20}$$

Ex ante, an agent seeks to maximize $U_n^0(a_{f,n}^0, a_n^0)$ as defined in (10)—and where $(a_{f,n}^{*,S}, a_n^{*,S})$ is given by (18)—subject to the budget constraint $(q_n^0)^\top p^0 + q_{f,n}^0 \leq W_n^0$. Given the assumption that the probability of each state S is the same under each model M, (10) simplifies to

$$U_n^0(a_{f,n}^0, a_n^0) = \mathrm{E}_S\left(U_n^S(a_{f,n}^{*,S}, a_n^{*,S}) \right).$$

Lemma A.6 shows that agents hold (a proportion of) the *market portfolio* as their uncertain portfolio in period 0 and agent n's market portfolio holding is proportional to his risk tolerance $\frac{1}{\theta_n}$. We know from our earlier analysis that interim period holdings will be exactly of the same form if agents are homogeneously ambiguity averse. However, with heterogeneity in ambiguity aversion we have trading over time, just as in Proposition 2.

PROPOSITION 4. *The equilibrium entails nontrivial trading if agents are heterogeneous in ambiguity aversion, that is, $\eta_1 \neq \eta_2$. If agents are homogeneous in ambiguity aversion, that is, $\eta_1 = \eta_2$, then the equilibrium entails no trade at any signal realization.*

Since agents hold the *market portfolio* in period 0, the ex ante portfolio holdings do not depend on the probabilities of the three interim states. The interim portfolio holdings are chosen after the state realizes, hence they do not depend on the state probabilities either. Therefore, trading volume does not depend on interim state probabilities. All the action that comes from these probabilities is subsumed by prices: as can be seen from (A.24) in Appendix A.2, the ex ante equilibrium price is a weighted sum of the interim equilibrium prices, where the weights are proportional to the probabilities of the states. For instance, if the probability of an interim state tends to 1, then the ex ante price converges to that state's price discounted by the risk-free rate. In this case, we will observe a small price change if the likely state were to arise. Furthermore, we know from (19) that the interim equilibrium prices vary with the signal realization S and are inversely related to $\bar{\Sigma}_S$, the level of ambiguity embodied by the realized signal. Therefore, if the level of ambiguity varies significantly across states and a state with small probability but relatively high ambiguity level were to arise, then there would be a significant drop in the asset prices. The price changes in this model are significant if and only if the state realizations are surprises and the ambiguity levels vary significantly across the states.

Next we relate the trading volume generated in our model to the level of ambiguity in the interim period. We focus on a scenario where the uncertainty shock does not affect all assets, in particular, it affects just asset 1, assumed to be more ambiguous (I) than asset 2. For a clearer statement of the result, we relabel the two agents in the economy as n and n'.

PROPOSITION 5. *Assume* $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, $\bar{\sigma}_1^H > \bar{\sigma}_1^I > \bar{\sigma}_1^L > \bar{\sigma}_2$, *and*

$$\bar{\Sigma}_S = \begin{pmatrix} \left(\bar{\sigma}_1^S\right)^2 & \bar{\sigma}_{12} \\ \bar{\sigma}_{12} & \left(\bar{\sigma}_2\right)^2 \end{pmatrix},$$

where $S = H, I, L$. *Let* $\eta_{n'} > \eta_n$ *and* Λ *be a constant equal to*

$$(\bar{\sigma}_2)^2 + 2\sqrt{(\bar{\sigma}_2)^4 + \frac{(\sigma^2 - \sigma_{12})^2 + (2 + \eta_n + \eta_{n'})(\sigma^2 - \sigma_{12})((\bar{\sigma}_2)^2 - \bar{\sigma}_{12}) + (1 + \eta_n)(1 + \eta_{n'})\bar{\sigma}_{12}(\bar{\sigma}_{12} - 2(\bar{\sigma}_2)^2)}{(1 + \eta_n)(1 + \eta_{n'})}}.$$

If

$$\left(\bar{\sigma}_1^S\right)^2 < \Lambda \quad \textit{for all } S, \tag{21}$$

then

$$\frac{a_{1,n}^{*,H}}{a_{2,n}^{*,H}} - \frac{a_{1,n'}^{*,H}}{a_{2,n'}^{*,H}} > \frac{a_{1,n}^{*,I}}{a_{2,n}^{*,I}} - \frac{a_{1,n'}^{*,I}}{a_{2,n'}^{*,I}} > \frac{a_{1,n}^{*,L}}{a_{2,n}^{*,L}} - \frac{a_{1,n'}^{*,L}}{a_{2,n'}^{*,L}} > 0.$$

This proposition posits that the higher the level of ambiguity following the uncertainty shock, *modulo the upper bound* (21), the bigger is the difference of the interim asset allocation ratios (i.e., the ratio of monies allocated to asset 1 vs. asset 2) between the less and the more ambiguity-averse agents. Given that all agents, ex ante, hold the market portfolio, for each agent the change in the asset allocation ratio across time is bigger following a shock resulting in a higher level of ambiguity. This implies that the dollar volume of trading is monotonically increasing in the level of ambiguity (uncertainty shock).

The conclusions we draw from Proposition 5 are tempered by the restriction on the ambiguity level given by (21). Why does this restriction arise? Recall from Corollary 1 that, if agent n' is more ambiguity averse than agent n and asset 1 is more ambiguous (I) than asset 2, then the difference in asset allocation ratios, $\frac{a_{1,n}^{*,S}}{a_{2,n}^{*,S}} - \frac{a_{1,n'}^{*,S}}{a_{2,n'}^{*,S}}$, is strictly positive for any given ambiguity level S. What Proposition 5 adds to that corollary is the implication about how these differences compare across the ambiguity levels. Notice though, the asset allocation ratios here are the ratios of monetary holdings, that is, price *times* quantity, whereas the corollary was true for quantity allocations, too. Asset prices differ across ambiguity levels (revealed through signal realizations); in particular, typically, the higher the ambiguity level of an asset the lower is that asset's price. Hence, it is possible that even if a higher ambiguity level leads to a bigger difference in the asset allocation ratio in quantities, the lower price attained in that ambiguity level would make the difference in monies smaller. The role of the restriction (21) is essentially to limit the ambiguity increase to a level that does not depress prices enough to reverse the monotonicity.

Dimmock et al. (2016) find empirical support for the notion that ambiguity aversion interacts with time-varying levels of economic uncertainty: in a representative U.S. household survey, conditional on holding stocks prior to the 2008–9 financial crisis, more ambiguity-averse households were more likely to actively reduce their equity holdings following the onset of the crisis. There are also empirical studies which make no reference to ambiguity aversion but find that investors change the composition of their uncertain portfolios in a particular way at times of high market volatility or ambiguity. Specifically, Giannetti and Laeven (2016) observe that investors rebalance their portfolios toward geographically close firms, which may be deemed as safer investments, during periods of high market volatility, whereas Kostopoulos et al. (2021) show that increases in ambiguity are positively associated with higher trading activity as well as individual (retail) investors reducing their exposure to the security market by trading out of stocks and similarly risky assets. All these empirical findings are in line with the main take-away from Proposition 5: the higher the level of ambiguity following an uncertainty shock, the higher is the dollar trading volume as more ambiguity-averse agents fly to safety, that is, less ambiguous assets. In addition, the novel insight in Proposition 4, that is, *nontrivial* trading following changes in ambiguity, speaks to the finding of Giannetti and Laeven (2016) showing that investors rebalance their portfolios toward stocks which may be deemed safer during times of market turmoil. Such behavior reflects nontrivial trading since this involves a change in the uncertain portfolio composition instead of trivial trading which is, simply, trading away from stocks to Treasuries.

To summarize the insights from the discussions in this section, trading volume is positively associated with the variation in ambiguity across periods whereas price changes are inversely related to the probability of the realized state (the extent to which the state is anticipated). Hence, even if the realized state is not a surprise, the level of trading volume can be significant because the resolution of ambiguity always changes the return–risk–ambiguity trade-off. If we assume that most announcements are not surprises, then, according to our model, nonnegligible trading volume with small price movements would be a common occurrence following announcements. However, on occasion a big surprise transpires, often accompanied by a relatively big change in ambiguity. In such a case, the big surprise would be associated with both a big price change and significant trading volume.

4.4. *Discussion: Robustness of Results.* This section discusses the robustness of our results on trade, in particular Proposition 2, to our modeling assumptions.

Robustness of the no-trade result with homogeneity in ambiguity aversion. Proposition 4 of Hara et al. (2022) implies that in the case of risk- and ambiguity-aversion specifications we have in this article if ambiguity attitudes are homogeneous then there is no trade in an interim period, irrespective of the common belief before and after the public signal,[26] and irrespective of the number of assets. The assertion extends to arbitrary number of interim periods as long as there is no intertemporal consumption, which is the case in our article. We outline the proof of the argument as follows: From Hara et al. (2022)'s Proposition 4, we know that all Pareto-optimal allocations are a linear share of the aggregate endowment whenever agents

[26] In particular, neither the first- nor the second-order uncertainty has to be Gaussian.

have homogeneous ambiguity aversion. In our framework, aggregate endowment is equal to the market portfolio since there are no nontradable endowments (agents are only endowed with tradable assets). Hence, markets are effectively complete, and so all equilibrium allocations, ex ante and interim, are Pareto-optimal allocations with agents holding a share of the market portfolio, negating the possibility of nontrivial trade. Actually, with some algebra, one may show that this assertion extends to trivial trade, implying that there is no trade, neither trivial nor nontrivial, under homogenous ambiguity aversion.[27]

Robustness of nontrivial trading result with heterogeneity in ambiguity aversion. In Subsection 4.2, we assume public signals, that is, earnings announcements, to be normally distributed mainly to simplify the Bayesian updating and thus retain tractability of dynamic analysis of the equilibrium. Our analysis in Subsection 4.3 shows that heterogeneity in ambiguity aversion gives rise to nontrivial trading even if the public signal is not normally distributed, albeit in a somewhat more stylized model compared to Subsection 4.2. We do not know if the nontrivial trading result generalizes beyond (recursive) robust mean–variance preferences. However, as we noted earlier, these preferences may be understood as approximations of recursive smooth ambiguity preferences with arbitrary utilities and beliefs (about asset returns).

5. DISCUSSION: OUR MECHANISM AND SUBJECTIVE BELIEFS

A way to understand how the mechanism of ambiguity aversion explains the key puzzles investigated in this article is through the construct of *as if* beliefs. Agent *n*'s as if beliefs are characterized by the product $\eta_n \Sigma_M$. The agent's ambiguity aversion, η_n, is in principle measurable from observations on the agent's choices over prospects not necessarily related to asset returns (e.g., choices from Ellsberg urns). The ambiguity of assets, $\sum_M$, can be estimated from public data on asset returns.

In Chiarella et al. (2010) and Gerber and Hens (2017), agents have mean–variance preferences with subjective beliefs about assets' return variances (and covariances), and it is shown that this leads to departures from the mutual fund theorem. A way to connect our theory to theirs would be through Equation (4). In their theory, the subjective belief is an agent-specific *primitive*, whereas for us, it is a *derived* object, $\eta_n \sum_M$. Hence, we are able to "unpack" the departure from the mutual fund theorem in terms of an agent-specific characteristic, η_n, and an objective characteristic that is specific to assets, $\sum_M$. Whereas, in their theory the departure is wholly characterized in terms of an agent-specific characteristic: each agent's subjective estimate of the covariance matrix of asset returns. Objective returns data do not put restrictions on the portfolio choice compatible with the subjective beliefs theory but it does for our theory.

It is natural to ask if our results and explanations could be replicated by *real* subjective beliefs with common objective means but different subjective variances across agents (as opposed to as if beliefs). First, subjective beliefs cannot explain the asset allocation puzzle, as argued by Canner et al. (1997), and therefore cannot connect this puzzle to the observed trade patterns following public signals. Second, as we have discussed, the literature which investigates trading volume driven by subjective beliefs relies on a speculation channel for which subjective beliefs with different means are necessary. Furthermore, in this literature the economies consist of one uncertain asset and one risk-free asset, and therefore any trade generated is (what we call) trivial trade.

It may also be useful to compare the empirical basis for our as if beliefs with that for real subjective beliefs which replicate the former's key features. As we have noted earlier, the difficulty of learning means from data justifies the assumption of ambiguity about mean returns. The other ingredient for as if beliefs, heterogeneity in attitudes toward ambiguity, is also well-documented in experimental studies. On the other hand, variances (and covariances) of future

[27] Proof of the assertion is available from the authors upon request. Note that Hara et al. (2022)'s analysis allows for a wider class of utility specifications.

returns are easier to estimate than means of future returns (see, e.g., Chan et al., 1999), given that the limit of infinitely fine sampling arguably removes all estimation risk. Hence, agents with subjective beliefs about variances should recognize, over time, that their beliefs are not consistent with the true variance because of the relative ease of learning variance from the realized data. Therefore, in the case of portfolio choice implications, any deviation from the mutual fund theorem due to subjective beliefs should eventually die out or at least should not remain significant for a sustained period of time.

However, one may argue that variances of returns are time-varying and stochastic, which in turn limits finer sampling opportunities and therefore makes estimation of second moments more difficult. Indeed, the macrofinance literature attempted to explain the dynamics of equity premium with stochastic volatility (more precisely, stochastic and time-varying covariance between aggregate consumption growth and market portfolio return). Even though this approach early on appeared to be compelling subsequent studies have found time variation/stochasticity in volatility to be economically insignificant or not sufficiently significant (Campbell, 2000, Lettau and Ludvigson, 2010 and Ludvigson, 2012), rendering the required subjective belief about stochasticity of volatility implausible. On the other hand, robustness/ambiguity-aversion modeling approach, which starts from the primitive of ambiguity about the mean of the aggregate consumption growth, yields as if beliefs in which the volatility of consumption growth is stochastic and has the right time variation to quantitatively match the observed dynamics of equity premium.[28]

6. CONCLUDING REMARKS

The main contribution of this article is to show that ambiguity aversion provides a unified framework to connect and explain seemingly distinct phenomena which constitute key empirical puzzles in the financial literature. The building block is the portfolio choice behavior under heterogeneous ambiguity aversion which can explain the asset allocation puzzle. This foundation delivers a distinct mechanism for understanding the nature of trade following public announcements, in particular, the significant trading volume that is accompanied by small price changes following earnings announcements and the positive association between trading activity and variations in ambiguity (i.e., ambiguity shocks). These phenomena can be seen as a consequence of nontrivial trading driven by differing ambiguity-sharing needs of heterogeneous ambiguity-averse agents responding to a commonly perceived change in ambiguity.

A natural item for future research is to explore whether ambiguity aversion can provide a testable theory which explains cross-sectional asset pricing puzzles such as beta, size, and value anomalies, among others. In our model, the systematic departure from the mutual fund theorem is that demands for more and less ambiguous assets differ across heterogeneously ambiguity-averse agents. This, of course, affects the equilibrium prices in the cross section of assets and hence implies a potential for systematic departure from the CAPM.

APPENDIX

A.1 *Parameter Values Used to Derive Figures 1 and 2.*

Parameterization of Figure 1. Following parameters reported in table 2 of Canner et al. (1997), we set $R_f = 1.06$, $\mu_{\text{bonds}} = 1.021$, $\mu_{\text{stocks}} = 1.09$, and

$$\Sigma_R = \begin{pmatrix} (\sigma_{\text{bonds}})^2 & \sigma_{\text{bonds,stocks}} \\ \sigma_{\text{bonds,stocks}} & (\sigma_{\text{stocks}})^2 \end{pmatrix} = \begin{pmatrix} (0.101)^2 & (0.23)(0.101)(0.208) \\ (0.23)(0.101)(0.208) & (0.208)^2 \end{pmatrix}.$$

[28] See Hansen and Sargent (2010), Ju and Miao (2012), Drechsler (2013), Bidder and Dew-Becker (2016), Collin-Dufresne et al. (2016), and especially Collard et al. (2018) where preferences are modeled in a similar fashion to our article.

We take

$$\Sigma_0 = \begin{pmatrix} (0.01)^2 & 0 \\ 0 & (0.158)^2 \end{pmatrix}$$

as the prior variance–covariance matrix for the means of bond and stock returns and combine it with the history of returns from 1926 to 1992. Then, the conjugate prior formula for the multivariate normal distribution gives us the model variance–covariance matrix as

$$\Sigma_M = \begin{pmatrix} \left(\sigma^M_{\text{bonds}}\right)^2 & \sigma^M_{\text{bonds,stocks}} \\ \sigma^M_{\text{bonds,stocks}} & \left(\sigma^M_{\text{stocks}}\right)^2 \end{pmatrix} = \begin{pmatrix} (0.00775)^2 & 0.000028 \\ 0.000028 & (0.02453)^2 \end{pmatrix}.$$

Our model specification, in particular (1), implies that $\Sigma_R > \Sigma_M$ and this condition is satisfied here. Also, the derived Σ_M implies a tight confidence interval around the unconditional means μ_{bonds} and μ_{stocks}, hence our numerical result for the optimal allocation ratio does not depend on relatively high levels of ambiguity. Finally, note that a wide range of priors can generate quantitatively comparable results.

Parameterization of Figure 2. The returns of the two uncertain assets are assumed to have the same mean and variance so that $M_1 = M_2$ and $\sigma_1^2 = \sigma_2^2$. The uncertainty about the mean (i.e., ambiguity) is assumed to be different across the assets so that $(\sigma_1^M)^2 \neq (\sigma_2^M)^2$. The parameter values for the risk-free asset and the uncertain assets are chosen based on the 1974–2015 nominal annual returns of the three-month U.S. T-Bills and the S&P500 index, respectively: R_f is rounded up as 1.05 from the sample mean of the three-month U.S. T-Bill (gross) returns, and $\sigma_1^2 = \sigma_2^2$ are rounded up as 0.04 from the sample variance of the S&P500 returns. We generate a return history for the uncertain assets by taking them to be the same as the 1974–2015 S&P500 index returns. Combining the noninformative prior over the mean with observations from this history, we find the variance of the resulting posterior distribution to be 0.0010. The parameters $(\sigma_1^M)^2 = 0.0006$ and $(\sigma_2^M)^2 = 0.0016$ are chosen around the latter figure so as to generate an economy with heterogeneously ambiguous assets. By rounding up the expected value of the posterior, we obtain $E[M_1] = E[M_2] = \mu_1 = \mu_2 = 1.12$ (which is a gross figure). We set $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$ and $\rho^M = \frac{\sigma_{12}^M}{\sigma_1^M\sigma_2^M}$ to be 0.5.

Following (1) and (13), conditional on the true means of the uncertain assets, the public signals (i.e., earnings announcements) are distributed such that $S|M \sim N(M, \Sigma_R - \Sigma_M)$. For our numerical exercise, we take the true means to be the same as the unconditional means, that is, we assume $M_1 = M_2 = 1.12$, and draw $1,000$ signal realizations from the distribution

$$N\left(\begin{pmatrix} 1.12 \\ 1.12 \end{pmatrix}, \begin{pmatrix} 0.04 - 0.0016 & (0.5)\sqrt{0.04}\sqrt{0.04} - (0.5)\sqrt{0.0006}\sqrt{0.0016} \\ (0.5)\sqrt{0.04}\sqrt{0.04} - (0.5)\sqrt{0.0006}\sqrt{0.0016} & 0.04 - 0.0006 \end{pmatrix}\right)$$

so as to simulate interim period returns and trading volumes generated by the signals. The economy is assumed to have four agents, who are heterogeneous in ambiguity aversion but homogeneous in risk aversion, with $\eta_1 = 0$, $\eta_2 = 3$, $\eta_3 = 9$, $\eta_4 = 12$, and $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$. The aggregate endowments of uncertain assets are normalized to 1. As is standard, asset i's trading volume is given by $\frac{\sum_n |q_{i,n}^S - q_{i,n}^0|}{2}$, where $q_{i,n}^S$ and $q_{i,n}^0$ are the quantities of asset i held by agent n after realization of signal S and prior to it, respectively.

A.2 *Proofs.*

LEMMA A.1. *Let agent n be more ambiguity averse than agent n', that is, $\eta_n > \eta_{n'}$. Then, for optimal portfolio choices it holds that*

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \text{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}},$$

if and only if

$$\left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb}\, S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 \;+\; \left(S_1^2 - S_2^2\right) \sigma_{12}^M\, \sigma_1^2\, \sigma_2^2$$
$$- \left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right) \sigma_{12} \left(\sigma_1^M\right)^2 \left(\sigma_2^M\right)^2 \;<\; 0. \tag{A.1}$$

PROOF OF LEMMA A.1. It follows from (8) that

$$\frac{\partial\left(\frac{a_{1,n}}{a_{2,n}}\right)}{\partial \eta_n} = \frac{C}{\left[(\mu_2 - R_f)\left(\sigma_1^2 + \eta_n(\sigma_1^M)^2\right) - (\mu_1 - R_f)\left(\sigma_{12} + \eta_n\sigma_{12}^M\right)\right]^2}, \tag{A.2}$$

where

$$\begin{aligned} C \;=\; &\left[(\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M\right]\left[(\mu_2 - R_f)(\sigma_1)^2 - (\mu_1 - R_f)\sigma_{12}\right] \\ &-\left[(\mu_2 - R_f)(\sigma_1^M)^2 - (\mu_1 - R_f)\sigma_{12}^M\right]\left[(\mu_1 - R_f)(\sigma_2)^2 - (\mu_2 - R_f)\sigma_{12}\right]. \end{aligned}$$

As $S_i = \frac{\mu_i - R_f}{\sigma_i}$ and $S_i^{Amb} = \frac{\mu_i - R_f}{\sigma_i^M}$, $i = 1, 2$, we can rewrite C as follows after some tedious but straightforward calculations:

$$C = \left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb} S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 + \left(S_1^2 - S_2^2\right)\sigma_{12}^M \sigma_1^2 \sigma_2^2$$
$$-\left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right)\sigma_{12}\left(\sigma_1^M\right)^2\left(\sigma_2^M\right)^2. \tag{A.3}$$

Therefore, following from (A.2) and (A.3), $\frac{\partial\left(\frac{a_{1,n}}{a_{2,n}}\right)}{\partial \eta_n} < 0$ if and only if

$$\left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb} S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 + \left(S_1^2 - S_2^2\right)\sigma_{12}^M \sigma_1^2 \sigma_2^2$$
$$-\left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right)\sigma_{12}\left(\sigma_1^M\right)^2\left(\sigma_2^M\right)^2 \;<\; 0. \tag{A.4}$$

Hence, $\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}}$ for $\eta_n > \eta_{n'}$ iff (A.4) holds. Since $\frac{a_{1,n}}{a_{2,n}} = \frac{p_1 q_{1,n}}{p_2 q_{2,n}}$ for all n, it also holds that $\frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}$ for $\eta_n > \eta_{n'}$ iff (A.4) holds. □

PROOF OF PROPOSITION 1.

(a) Let $S_1 = S_2$. Then the left-hand side of (A.1) reduces to

$$\left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right)\left(\sigma_1^M\right)^2\left(\sigma_2^M\right)^2\sigma_1\sigma_2(1-\rho),$$

which is strictly less than 0 if $\rho \neq 1$ and $S_1^{Amb} < S_2^{Amb}$. Hence, the desired result follows from Lemma A.1.

(b) After somewhat tedious but routine calculations the left-hand side of (A.1) can be rewritten as

$$(\mu_1 - R_f)(\mu_2 - R_f) \times$$
$$\left[\left(\sigma_1^2(\sigma_2^M)^2 - \sigma_2^2(\sigma_1^M)^2\right) + \frac{\mu_1 - R_f}{\mu_2 - R_f}\left(\sigma_2^2\sigma_{12}^M - (\sigma_2^M)^2\sigma_{12}\right) + \frac{\mu_2 - R_f}{\mu_1 - R_f}\left((\sigma_1^M)^2\sigma_{12} - \sigma_1^2\sigma_{12}^M\right)\right]. \tag{A.5}$$

Assume that $\rho(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}) \leq \rho^M \leq \rho(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2})$. Then the expression in (A.5), and therefore the left-hand side of (A.1), is less than or equal to

$$\begin{aligned}
&(\mu_1 - R_f)(\mu_2 - R_f)\left(\sigma_1^2(\sigma_2^M)^2 - \sigma_2^2(\sigma_1^M)^2\right) \\
&= (\mu_1 - R_f)(\mu_2 - R_f)\sigma_2^2(\sigma_2^M)^2\left(\left(\frac{\sigma_1}{\sigma_2}\right)^2 - \left(\frac{\sigma_1^M}{\sigma_2^M}\right)^2\right).
\end{aligned}$$

The above expression is strictly less than 0 if $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$ (which is equivalent to $\sigma_1^M/\sigma_2^M > \sigma_1/\sigma_2$). Therefore, the desired result follows from Lemma A.1. □

PROOF OF COROLLARY 1. This is a direct corollary of Proposition 1. □

PROOF OF COROLLARY 2. Observe from (8) that

$$\lim_{\eta_n \to \infty} \frac{a_{1,n}}{a_{2,n}} = \frac{(\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M}{(\mu_2 - R_f)(\sigma_1^M)^2 - (\mu_1 - R_f)\sigma_{12}^M}.$$

If $S_1^{Amb} < S_2^{Amb}$ and $\sigma_1^M > \sigma_2^M$, then

$$\frac{\mu_1 - R_f}{\mu_2 - R_f} < \frac{\sigma_1^M}{\sigma_2^M} \leq \left(\frac{\sigma_1^M}{\sigma_2^M}\right)\left(\frac{\sigma_1^M + \rho^M\sigma_2^M}{\sigma_2^M + \rho^M\sigma_1^M}\right) = \frac{(\sigma_1^M)^2 + \sigma_{12}^M}{(\sigma_2^M)^2 + \sigma_{12}^M}.$$

This further implies that $\lim_{\eta_n \to \infty} \frac{a_{1,n}}{a_{2,n}} < 1$. □

LEMMA A.2. *Optimal interim monetary holdings of agent n are given by*

$$a_n^S = \frac{1}{\theta_n}(A_n + B_n S), \tag{A.6}$$

where

$$\begin{aligned}
A_n &= (\Sigma + (\eta_n + 1)\Sigma_S)^{-1}\left[\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma_M^{-1}\mu - R_f 1\right], \\
B_n &= (\Sigma + (\eta_n + 1)\Sigma_S)^{-1}\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma^{-1}.
\end{aligned}$$

Furthermore, B_n is a symmetric matrix.

PROOF OF LEMMA A.2. Recall from (21), the interim maximization problem of agent n reduces to

$$\max_{a_n^S}\left\{\Big(W_n^S(a_{n,f}^0,a_n^0)-(a_n^S)1\Big)R_f+(a_n^S)^\top\mu_S-\frac{\theta_n}{2}(a_n^S)^\top(\Sigma+\Sigma_S)a_n^S-\frac{\gamma_n-\theta_n}{2}(a_n^S)^\top\Sigma_S a_n^S\right\}.$$

First-order condition for this problem yields

$$-R_f1+\mu_S-\theta_n(\Sigma+\Sigma_S)a_n^S-(\gamma_n-\theta_n)\Sigma_S a_n^S=0.$$

Since $\eta_n=\frac{\gamma_n-\theta_n}{\theta_n}$,

$$-R_f1+\mu_S-\theta_n(\Sigma+\Sigma_S)a_n^S-\theta_n\eta_n\Sigma_S a_n^S=0$$

and thus

$$a_n^S=\frac{1}{\theta_n}(\Sigma+\Sigma_S+\eta_n\Sigma_S)^{-1}(\mu_S-R_f1)=\frac{1}{\theta_n}(\Sigma+(\eta_n+1)\Sigma_S)^{-1}(\mu_S-R_f1)$$

or, recalling that $\Sigma_{R'}=\Sigma+\Sigma_S$,

$$a_n^S=\frac{1}{\theta_n}(\Sigma_{R'}+\eta_n\Sigma_S)^{-1}(\mu_S-R_f1).$$

Note that μ_S is a function of S, whereas Σ_S and Σ (and thus $\Sigma_{R'}$) are not. Plugging the expression for $\mu_S=(\Sigma_M^{-1}+\Sigma^{-1})^{-1}(\Sigma_M^{-1}\mu+\Sigma^{-1}S)$, we obtain:

$$a_n^S=\frac{1}{\theta_n}(A_n+B_nS),$$

where

$$\begin{aligned}A_n&=(\Sigma+(\eta_n+1)\Sigma_S)^{-1}\Big[\big(\Sigma_M^{-1}+\Sigma^{-1}\big)^{-1}\Sigma_M^{-1}\mu-R_f1\Big],\\B_n&=(\Sigma+(\eta_n+1)\Sigma_S)^{-1}\big(\Sigma_M^{-1}+\Sigma^{-1}\big)^{-1}\Sigma^{-1}.\end{aligned}$$

Also, observe that

$$\begin{aligned}B_n&=(\Sigma+(\eta_n+1)\Sigma_S)^{-1}\big(\Sigma_M^{-1}+\Sigma^{-1}\big)^{-1}\Sigma^{-1}\\&=(\Sigma+(\eta_n+1)\Sigma_S)^{-1}\big(\Sigma\big(\Sigma_M^{-1}+\Sigma^{-1}\big)\big)^{-1}\\&=\big[\big(\Sigma\Sigma_M^{-1}+I\big)(\Sigma+(\eta_n+1)\Sigma_S)\big]^{-1}\\&=\left[\Sigma\Sigma_M^{-1}\Sigma+\Sigma+(\eta_n+1)\Sigma\underbrace{\big(\Sigma_S^{-1}-\Sigma^{-1}\big)}_{\text{follows from Eqn. (15)}}\Sigma_S+(\eta_n+1)\Sigma_S\right]^{-1}\\&=\big[\Sigma\Sigma_M^{-1}\Sigma+\Sigma+(\eta_n+1)\Sigma\big]^{-1},\end{aligned}$$

which is symmetric since $\Sigma\Sigma_M^{-1}\Sigma$ and Σ are both symmetric. □

LEMMA A.3. *At interim equilibrium, for any $S \in \mathbb{R}$, $i \in \{1,2\}$ and $n \in \{1,\dots,N\}$, agent n's holding of asset i is given by:*

$$q_{i,n}^S = \begin{cases} \frac{\frac{1}{\theta_n}}{\sum_k \frac{1}{\theta_k}} \times e & \text{if} \quad \eta_n = \eta_k \quad \forall k \in \{1,\dots,N\} \\ \frac{\frac{1}{\theta_n}(A_n + B_n S)_i}{\sum_k \frac{1}{\theta_k}(A_k + B_k S)_i} \times e & \text{otherwise,} \end{cases} \tag{A.7}$$

where A_n and B_n are as defined in Lemma A.2, and

$$p^S = A + BS, \tag{A.8}$$

where $A \equiv \frac{1}{e}\sum_n \frac{1}{\theta_n} A_n$ and $B \equiv \frac{1}{e}\sum_n \frac{1}{\theta_n} B_n$.[29]

PROOF OF LEMMA A.3 . This lemma follows from Lemma A.2 and market clearing. □

LEMMA A.4. *At ex ante equilibrium, for any $i \in \{1,2\}$ and $k, n \in \{1,\dots,N\}$, the ratio of agents k and n's asset i holdings is given by:*

$$\frac{q_{i,k}^0}{q_{i,n}^0} = \begin{cases} \frac{\theta_n}{\theta_k} & \text{if} \quad \eta_n = \eta_k \\ \frac{\theta_n}{\theta_k} \frac{\left(Q(\eta_k, p^0)\right)_i}{\left(Q(\eta_n, p^0)\right)_i} & \text{otherwise,} \end{cases}$$

where $Q(\eta_n, p^0)$ is a vector whose expression is as given by (A.10) in the proof below.

PROOF OF LEMMA A.4 . First note that, for given monetary holdings $(a_{f,n}^0, a_n^0)$:

$$\begin{aligned} &\mathrm{E}_{R'|M'}\Big[u_n\Big(W_n^2(a_{f,s}^S, a_n^S)\Big)\Big] = \\ &= \mathrm{E}_{R'|M'}\Big[u_n\Big((a_n^S)^\top R + \big(W_n^0 - (q_n^0)^\top p^0\big)R_f^2 + \big((q_n^0)^\top p^S - (a_n^S)^\top 1\big)R_f\Big)\Big] \\ &= -\exp\bigg(-\theta_n\bigg[\big(W_n^0 - (q_n^0)^\top p^0\big)R_f^2 + \big((q_n^0)^\top p^S - (a_n^S)^\top 1\big)R_f + (a_n^S)^\top M' - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_S^n\bigg]\bigg). \end{aligned}$$

Hence, $\mathrm{E}_{M'}[\phi_n(\mathrm{E}_{R'|M'}[u_n(W_n^2(a_{f,s}^S, a_n^S))])]$ equals

$$\begin{aligned} &-\exp\bigg(-\gamma_n\bigg[\big(W_n^0 - (q_n^0)^\top p^0\big)R_f^2 + \big((q_n^0)^\top p^S - (a_n^S)^\top 1\big)R_f - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_n^S\bigg]\bigg) \times \\ &\exp\bigg(-\gamma_n (a_n^S)^\top \mu_S + \frac{\gamma_n^2}{2}(a_n^S)^\top \Sigma_S a_n^S\bigg). \end{aligned}$$

Since $\phi_n^{-1}(z) = -(-z)^{\frac{\theta_n}{\gamma_n}}$, we have

$$\begin{aligned} U_n^S(a_{f,s}^S, a_n^S) &= \phi_n^{-1}\Big(\mathrm{E}_{M'}\Big[\phi_n\Big(\mathrm{E}_{R'|M'}\Big[u_n\Big(W_n^2(a_{f,s}^S, a_n^S)\Big)\Big]\Big)\Big]\Big) \\ &= -\exp\bigg(-\theta_n\bigg[\big(W_n^0 - (q_n^0)^\top p^0\big)R_f^2 + \big((q_n^0)^\top p^S - (a_n^S)^\top 1\big)R_f + (a_n^S)^\top \mu_S \\ &\quad - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_n^S - \frac{\gamma_n}{2}(a_n^S)^\top \Sigma_S a_n^S\bigg]\bigg). \end{aligned}$$

[29] For a vector V, we to denote the ith component of the vector by $(V)_i$.

Plugging in a_n^S and p^S from (A.6) and (A.8), and using the fact that $\mu_S = (\Sigma_M^{-1} + \Sigma^{-1})^{-1} \times (\Sigma_M^{-1}\mu + \Sigma^{-1}S)$, we obtain:

$$U_n^S(a_{f,s}^{*,S}, a_n^{*,S}) = -exp\Bigg(-\theta_n\Bigg[\big(W_n^0 - (q_n^0)^\top p^0\big)R_f^2 + (q_n^0)^\top(A + BS)R_f + \frac{1}{\theta_n}(A_n + B_nS)^\top(\mu_S - 1R_f)$$
$$-\frac{1}{2\theta_n}(A_n + B_nS)^\top\Sigma(A_n + B_nS) - \frac{\gamma_n}{2\theta_n^2}(A_n + B_nS)^\top\Sigma_S(A_n + B_nS)\Bigg]\Bigg)$$
$$= -\Gamma_n\, exp\big(S^\top C_nS + d_n^\top S + e_n\big),$$

where $\Gamma_n > 0$ does not depend on q_n^0, S, or M, and

$$C_n = \frac{1}{2}B_n^\top\Sigma B_n + \frac{1}{2}(\eta_n + 1)B_n^\top\Sigma_SB_n - \Big(\big(\Sigma_M^{-1} + \Sigma^{-1}\big)^{-1}\Sigma^{-1}\Big)^\top\big(\Sigma + (\eta_n + 1)\Sigma_S\big)^{-1}\big(\Sigma_M^{-1} + \Sigma^{-1}\big)^{-1}\Sigma^{-1},$$
$$d_n^\top = -\theta_n(q_n^0)^\top BR_f - A_n^\top\Sigma_S\Sigma^{-1} - \big(\Sigma_S\Sigma_M^{-1}\mu - 1R_f\big)^\top B_n + A_n^\top\Sigma B_n + (\eta_n + 1)A_n^\top\Sigma_SB_n,$$
$$e_n = -\theta_n(q_n^0)^\top AR_f + \theta_n(q_n^0)^\top p^0R_f^2.$$

Note that we rely on the fact that B_n is symmetric and on its expression given in Lemma A.2 in the derivation of C_n. Next, letting $Y \equiv S - M$ yields

$$U_n^S(a_{f,s}^{*,S}, a_n^{*,S}) = -\Gamma_n\, \exp\big(Y^\top C_nY + (d_n^\top + 2M^\top C_n)Y + e_n + M^\top C_nM + {d_n}^\top M\big).$$

We now introduce a well-known result about multivariate normal distributions (e.g., see Anderson (1984, Ch. 2) or Brunnermeier (2001, p. 64)):

Mathematical Preliminary . *Let* $\omega \sim N(0, \bar{\Sigma})$. *Then,*

$$E\left[\exp\left(\omega^\top\bar{A}\omega + \bar{b}^\top\omega + \bar{c}\right)\right] = |I - 2\bar{\Sigma}\bar{A}|^{-1/2}\exp\left(\frac{1}{2}\bar{b}^\top\left(I - 2\bar{\Sigma}\bar{A}\right)^{-1}\bar{\Sigma}\bar{b} + \bar{c}\right),$$

where $\bar{A}$ *is a symmetric matrix*, $\bar{b}$ *a vector, and* $\bar{c}$*a scalar*.

Note that $Y|M \sim N(0, \Sigma)$. Also, observe that C_n is a symmetric matrix. Therefore, using the Mathematical Preliminary, we get the following:

$$\mathrm{E}_{S|M}\Big[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\Big] = -\Gamma_n\, \mathrm{E}_{Y|M}\big[\exp\big(Y_n^\top C_nY + ({d_n}^\top + 2M^\top C_n)Y + e_n + M^\top C_nM + {d_n}^\top M\big)\big]$$
$$= -\Gamma_n\, |I - 2\Sigma C_n|^{-1/2}\, \exp\big(M^\top D_nM + f_n^\top M + g_n\big),$$

where $D_n = 2(C_n^{-1}\Sigma^{-1}C_n^{-1} - 2C_n^{-1})^{-1} + C_n$, $f_n^\top = 2d_n^\top(I - 2\Sigma C_n)^{-1}\Sigma C_n + d_n^\top$, and $g_n = \frac{1}{2}{d_n}^\top(I - 2\Sigma C_n)^{-1}\Sigma d_n + e_n$. Note, D_n is symmetric. Making use of the Mathematical Preliminary once again, we get:

$$\phi_n\Big(U_n^0\Big(a_{f,n}^0, a_n^0\Big)\Big) = \mathrm{E}_M\Big[\phi_n\Big(\mathrm{E}_{S|M}\Big[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\Big]\Big)\Big]$$
$$= -\Gamma_n^{\gamma_n/\theta_n}\, |I - 2\Sigma C_n|^{-\gamma_n/2\theta_n}\, \mathrm{E}_M\Big[\exp\Big(\frac{\gamma_n}{\theta_n}\big[M^\top D_nM + f_n^\top M + g_n\big]\Big)\Big]$$
$$= -\Gamma_n^{\gamma_n/\theta_n}\, |I - 2\Sigma C_n|^{-\gamma_n/2\theta_n}\, |I - 2\Sigma_M\frac{\gamma_n}{\theta_n}D_n|^{-1/2} \times$$
$$\exp\Big(\frac{1}{2}\frac{\gamma_n^2}{\theta_n^2}(f_n^\top + 2\mu^\top D_n)(I - 2\Sigma_M\frac{\gamma_n}{\theta_n}D_n)^{-1}\Sigma_M(f_n^\top + 2\mu^\top D_n)^\top + \frac{\gamma_n}{\theta_n}(g_n + f_n^\top\mu + \mu^\top D_n\mu)\Big).$$

Since agent n maximizes over q_n^0 (which is equivalent to maximizing over a_n^0 given prices), we can focus only on elements containing q_n^0 in the objective function. Therefore,

$$\begin{aligned}&\arg\max_{q_n^0} \mathrm{E}_M\Big[\phi_n\Big(\mathrm{E}_S\Big[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\Big]\Big)\Big] = \\ &\arg\max_{q_n^0} \left\{-\frac{1}{2}\frac{\gamma_n}{\theta_n}(f_n^\top + 2\mu^\top D_n)E_n(f_n^\top + 2\mu^\top D_n)^\top - g_n - f_n^\top \mu\right\},\end{aligned}$$

where $E_n = (I - 2\Sigma_M \frac{\gamma_n}{\theta_n} D_n)^{-1}\Sigma_M = (\Sigma_M^{-1} - 2(\eta_n + 1)D_n)^{-1}$, which depends only on η_n and not on θ_n. Next, let

$$\begin{aligned}F_n &= (I - 2\Sigma C_n)^{-1}\Sigma, \\ G_n &= -A_n^\top \Sigma_S \Sigma^{-1} - \left(\Sigma_S \Sigma_M^{-1}\mu - 1R_f\right)^\top B_n + A_n^\top \Sigma B_n + (\eta_n + 1)A_n^\top \Sigma_S B_n.\end{aligned}$$

Note, both F_n and G_n depend only on η_n and not on θ_n. Observe that

$$\begin{aligned}&\arg\max_{q_n^0} \mathrm{E}_M\Big[\phi_n \mathrm{E}_S\Big[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\Big]\Big] = \\ &\arg\max_{q_n^0} \Big\{-\frac{1}{2}\gamma_n\theta_n(R_f)^2(q_n^0)^\top B(2F_nC_n + I)E_n(2F_nC_n + I)^\top B^\top q_n^0 \\ &+2\gamma_n R_f(q_n^0)^\top B(2F_nC_n + I)E_n\mu_n^\top D_n + \theta_n R_f(q_n^0)^\top BF_nG_n \\ &-\frac{1}{2}\theta_n^2 R_f^2(q_n^0)^\top BF_nB^\top q_n^0 + \theta_n(q_n^0)^\top(p^0R_f^2 - AR_f) + \theta_n R_f(q_n^0)^\top B(2F_nC_n + I)\mu\Big\}.\end{aligned}$$

The first-order condition for the above maximization problem yields:

$$\begin{aligned}0 = &-\gamma_n\theta_n R_f B(2F_nC_n + I)E_n(2F_nC_n + I)^\top B^T q_n^0 + 2\gamma_n B(2F_nC_n + I)E_n\mu^\top D_n + \theta_n BF_nG_n \\ &-\theta_n^2 R_f BF_nB^\top q_n^0 + \theta_n(p^0R_f - A) + \theta_n B(2F_nC_n + I)\mu.\end{aligned}$$

This implies that

$$q_n^0 = \frac{1}{\theta_n R_f} Q(\eta_n, p^0), \tag{A.9}$$

where

$$\begin{aligned}Q(\eta_n, p^0) = &\left((\eta_n + 1)B(2F_nC_n + I)E_n(2F_nC_n + I)^\top B^\top + BF_nB^\top\right)^{-1} \times \\ &\left(2(\eta_n + 1)B(2F_nC_n + I)E_n\mu^\top D_n + BF_nG_n - A + B(2F_nC_n + I)\mu + p^0R_f\right). \quad \text{(A.10)}\end{aligned}$$

Observe that the vector $Q(\eta_n, p^0)$ depends neither on θ_n nor on S. Taking the ratio of ex ante equilibrium asset i holdings of agents k and n given by (A.9), we get the desired result. □

LEMMA A.5. *At ex ante equilibrium, for any $i \in \{1, 2\}$ and $n \in \{1, \ldots, N\}$, agent n's holding of asset i is*

$$q_{i,n}^0 = \begin{cases}\frac{\frac{1}{\theta_n}}{\sum_n \frac{1}{\theta_n}} \times e & \text{if} \quad \eta_n = \eta_k \quad \forall k \in \{1, \ldots, N\} \\ \frac{1}{\theta_n R_f} \times (Q^\star(\eta_n))_i & \text{otherwise,}\end{cases}$$

where $Q^\star(\eta_n)$ is a vector whose expression is as given by (A.12) in the proof below.

PROOF OF LEMMA A.5 . Rewrite the vector $Q(\eta_n, p^0)$ given by A.10) as

$$Q(\eta_n, p^0) = (Y_n)^{-1}(Z_n + p^0 R_f), \tag{A.11}$$

where

$$Y_n = (\eta_n + 1)B(2F_nC_n + I)E_n(2F_nC_n + I)^\top B^\top + BF_nB^\top,$$
$$Z_n = 2(\eta_n + 1)B(2F_nC_n + I)E_n\mu^\top D_n + BF_nG_n - A + B(2F_nC_n + I)\mu.$$

The market clearing condition implies that $\sum_k \frac{1}{\theta_k}(Y_k)^{-1}(Z_k + p^0 R_f) \;=\; R_f\, e$. Hence, at an ex ante equilibrium, it holds that

$$p^0\, R_f \;=\; \left(\sum_k \frac{1}{\theta_k}(Y_k)^{-1}\right)^{-1}\left(R_f\, e - \sum_k \frac{1}{\theta_k}(Y_k)^{-1}Z_k\right).$$

Plugging this back into (A.11) yields

$$Q^\star(\eta_n) = (Y_n)^{-1}\left[Z_n + \left(\sum_k \frac{1}{\theta_k}(Y_k)^{-1}\right)^{-1}\left(R_f\, e - \sum_k \frac{1}{\theta_k}(Y_k)^{-1}Z_k\right)\right]. \tag{A.12}$$

Hence, it follows from (A.9) that

$$q^0_{i,n} \;=\; \frac{1}{\theta_n R_f} \times (Q^\star(\eta_n))_i. \tag{A.13}$$

Under homogeneity of ambiguity aversion, we know from Lemma A.4 that $q^0_{i,k} = q^0_{i,1}\,\frac{\theta_1}{\theta_k}$ for all $k \in \{1, \ldots, N\}$. Therefore, for any $i = 1, 2$, it follows from the market clearing condition that $q^0_{i,1}\,\theta_1 \sum_k \frac{1}{\theta_k} \;=\; e$. This implies that $q^0_{i,1} \;=\; \frac{e}{\theta_1}\,\frac{1}{\sum_k \frac{1}{\theta_k}}$, which in turn implies

$$q^0_{i,n} \;=\; \frac{\frac{1}{\theta_n}}{\sum_k \frac{1}{\theta_k}} \times e \tag{A.14}$$

under homogeneous ambiguity aversion. (A.13) and (A.14) together yield the desired result. □

PROOF OF PROPOSITION 2. Under homogeneity of ambiguity aversion, Lemmas A.3 and A.5 establish that agents' asset holdings are the same ex ante and interim regardless of the signal realization. Under heterogeneous ambiguity aversion, observe from Lemmas A.3 and A.5 that $\frac{q^S_{1,n}}{q^S_{2,n}}$ depends on the realization of S whereas $\frac{q^0_{1,n}}{q^0_{2,n}}$, does not. □

PROOF OF PROPOSITION 3. We know that, upon observing signal S, the interim period equilibrium asset price vector is given by $p^S = \frac{\sum_n \frac{1}{\theta_n}(A_n + B_nS)}{e} \equiv A + BS$. Since B, a symmetric matrix, is generically invertible, there generically exists a signal realization, call it $\check{S}$, such that the interim period equilibrium asset price vector, p^S, is equal to the initial period equilibrium asset price vector, p^0: $\check{S} = B^{-1}(p^0 - A)$.

It follows from Lemma A.2 that

$$q^{\check{S}}_{i,n} = \frac{a^{\check{S}}_{i,n}}{p^{\check{S}}_i} = \frac{\frac{1}{\theta_n}\big(A_n + B_nB^{-1}(p^0 - A)\big)_i}{\sum_n \frac{1}{\theta_n}(A_n + B_nB^{-1}(p^0 - A))_i} \times e, \quad i = 1, 2,$$

and therefore

$$\text{(A.15)} \qquad \frac{q^{\check{S}}_{1,n}}{q^{\check{S}}_{2,n}} = \frac{\big(A_n + B_n B^{-1}\big(p^0 - A\big)\big)_1}{\big(A_n + B_n B^{-1}(p^0 - A)\big)_2} \times \frac{\sum_n \frac{1}{\theta_n}\big(A_n + B_n B^{-1}\big(p^0 - A\big)\big)_2}{\sum_n \frac{1}{\theta_n}\big(A_n + B_n B^{-1}(p^0 - A)\big)_1}.$$

Also, from (A.9), we know that

$$\text{(A.16)} \qquad \frac{q^0_{1,n}}{q^0_{2,n}} = \frac{(Q(\eta_n, p^0))_1}{(Q(\eta_n, p^0))_2},$$

where $Q(\eta_n, p^0)$ is as defined in (A.10).

Let

$$f\big(\{\theta_k\}_{k=1}^N, \eta_n, p^0\big) \equiv \frac{\big(A_n + B_n B^{-1}\big(p^0 - A\big)\big)_1}{\big(A_n + B_n B^{-1}(p^0 - A)\big)_2} \times \frac{\sum_n \frac{1}{\theta_n}\big(A_n + B_n B^{-1}\big(p^0 - A\big)\big)_2}{\sum_n \frac{1}{\theta_n}\big(A_n + B_n B^{-1}(p^0 - A)\big)_1},$$

$$g(\eta_n, p^0) \equiv \frac{(Q(\eta_n, p^0))_1}{(Q(\eta_n, p^0))_2}.$$

Following (A.15) and (A.16), the equilibrium does not entail nontrivial trading at $\check{S}$ if and only if

$$\text{(A.17)} \qquad f\big(\{\theta_k\}_{k=1}^N, \eta_n, p^0\big) = g(\eta_n, p^0) \quad \text{for all } n.$$

Note that f generically depends on $\{\theta_k\}_{k=1}^N$ whereas g does not, therefore

$$F\big(\big(\{\theta_k\}_{k=1}^N, \eta_n, \cdot\big)\big) \equiv f\big(\{\theta_k\}_{k=1}^N, \eta_n, \cdot\big) - g(\eta_n, \cdot)$$

is a nondegenerate function of $(\{\theta_k\})_{k=1}^N$ and η_n. Let $p(\{\theta_k\}_{k=1}^N, \eta_n)$ solve

$$F\big(\{\theta_k\}_{k=1}^N, \eta_n, \cdot\big) = 0.$$

Then $p((\{\theta_k\}_{k=1}^N, \eta_n)$ generically depends on η_n, implying that $p(\{\theta_k\}_{k=1}^N, \eta_n) \neq p(\{\theta_k\}_{k=1}^N, \eta_{n'})$ if $\eta_n \neq \eta_{n'}$. This, in turn, implies that if $f(\{\theta_k\}_{k=1}^N, \eta_n, p^0) = g(\eta_n, p^0)$ then $f(\{\theta_k\}_{k=1}^N, \eta_{n'}, p^0) \neq g(\eta_{n'}, p^0)$ for $n \neq n'$. This violates the condition given in (A.17) and therefore we have the desired result. □

LEMMA A.6. *At ex ante equilibrium, for any $i \in \{1, 2\}$ and $n \in \{1, 2\}$, agent n's holding of asset i is*

$$q^0_{i,n} = \frac{\frac{1}{\theta_n}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \times e.$$

PROOF OF LEMMA A.6. Given monetary holdings $(a^S_{f,n}, a^S_n)$, we have

$$\begin{aligned}
&E_{R'|M'}\Big[u_n(W_n^2(a^S_{f,n}, a^S_n))\Big] = \\
&= E_{R'|M'}[-\exp(-\theta_n W_n^2(a^S_{f,n}, a^S_n))] \\
&= -\exp\Big(-\theta_n\Big[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\Big]\Big)\, E_{R'|M'}\big[\exp\big(-\theta_n (a_n^S)^\top R\big)\big]
\end{aligned}$$

$$= -\exp\Big(-\theta_n\Big[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\Big]\Big) \times$$
$$\exp\bigg(-\theta_n (a_n^S)^\top E_{R'|M'}[R] + \frac{1}{2}\theta_n^2 (a_n^S)^\top \mathrm{var}_{R'|M'}(R) a_n^S\bigg)$$
$$= -\exp\Big(-\theta_n\Big[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\Big]\Big)\exp\bigg(-\theta_n (a_n^S)^\top \mu + \frac{1}{2}\theta_n^2 (a_n^S)^\top \Sigma a_n^S\bigg).$$

Then,

$$E_{M'}\Big(\phi_n\Big[E_{R'|M'}\Big[u_n(W_n^2(a_{f,n}^S, a_n^S))\Big]\Big]\Big) =$$
$$= -\exp\Big(-\gamma_n\Big\{(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\Big\}\Big)\exp\bigg(-\gamma_n (a_n^S)^\top \mu + \frac{\gamma_n\theta_n}{2}(a_n^S)^\top \bar{\Sigma} a_n^S\bigg),$$

which in turn yields

$$U_n^S(a_{f,n}^S, a_n^S) = \phi_n^{-1}\Big(E_{M'}\Big(\phi_n\Big[E_{R'|M'}\Big[u_n(W_n^2(a_{f,n}^S, a_n^S))\Big]\Big]\Big)\Big) =$$
$$= -\exp\bigg(-\theta_n\bigg\{\Big(W_n^0 - (q_n^0)^\top p^0\Big)R_f^2 + \Big((q_n^0)^\top p^S - (a_n^S)^\top 1\Big)R_f + (a_n^S)^\top \mu - \frac{1}{2}\gamma_n (a_n^S)^\top \Sigma a_n^S\bigg\}\bigg)$$
$$= K^S \exp\Big(-\theta_n\Big\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big), \tag{A.18}$$

where

$$K^S = -\exp\bigg(-\theta_n\bigg\{(W_n^0 R_f - (a_n^S)^\top 1)R_f + (a_n^S)^\top \mu - \frac{1}{2}\gamma_n (a_n^S)^\top \Sigma a_n^S\bigg\}\bigg).$$

Note, K^S does not depend on q_n^0. Following (A.18), agent n's maximization problem reduces to:

$$\max_{q_n^0} \sum_{S\in\{H,I,L\}} \pi(S)\, K^S\, exp\Big(-\theta_n\Big\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big).$$

The first-order condition of this problem is

$$0 = \pi(H)\, K^H\, (p_i^H - p_i^0 R_f)R_f \exp\Big(-\theta_n\Big\{(q_n^0)^\top p^H R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big)$$
$$+ \pi(I)\, K^I\, (p_i^I - p_i^0 R_f)R_f \exp\Big(-\theta_n\Big\{(q_n^0)^\top p^I R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big)$$
$$+ \pi(L)\, K^L\, (p_i^L - p_i^0 R_f)R_f \exp\Big(-\theta_n\Big\{(q_n^0)^\top p^L R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big), \quad i = 1, 2.$$

Let

$$A^S = \exp\Big(-\theta_n\Big\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\Big\}\Big), \quad S = H, I, L. \tag{A.19}$$

Then, the first-order condition above can be rewritten as

$$0 = \pi(H)K^H(p_i^H - p_i^0 R_f)R_f A^H + \pi(I)K^I(p_i^I - p_i^0 R_f)R_f A^I + \pi(L)K^L(p_i^L - p_i^0 R_f)R_f A^L, \quad i = 1, 2,$$

which implies

$$\text{(A.20)} \qquad \frac{A^H}{A^I} = \frac{K^I\pi(I)\big(R_f p_2^0(p_1^L - p_1^I) + p_2^L p_1^I - p_1^L p_2^I + R_f p_1^0\big(p_2^I - p_2^L\big)\big)}{K^H\pi(H)\big(R_f p_2^0(p_1^H - p_1^L) + p_2^H p_1^L - p_1^H p_2^L + R_f p_1^0\big(p_2^L - p_2^H\big)\big)},$$

$$\text{(A.21)} \qquad \frac{A^L}{A^I} = \frac{K^I\pi(I)\big(R_f p_2^0(p_1^I - p_1^H) + p_1^H p_2^I - p_2^H p_1^I + R_f p_1^0\big(p_2^H - p_2^I\big)\big)}{K^L\pi(L)\big(R_f p_2^0(p_1^H - p_1^L) + p_2^H p_1^L - p_1^H p_2^L + R_f p_1^0\big(p_2^L - p_2^H\big)\big)}.$$

Also, it follows from (A.19) thatvv

$$\frac{A^H}{A^I} = \exp\big(-\theta_n\big[q_{1,n}^0(p_1^H - p_1^I) + q_{2,n}^0(p_2^H - p_2^I)R_f\big]\big),$$
$$\frac{A^L}{A^I} = \exp\big(-\theta_n\big[q_{1,n}^0(p_1^L - p_1^I) + q_{2,n}^0(p_2^L - p_2^I)R_f\big]\big).$$

We have two equations with two unknowns, namely, $q_{1,n}^0$ and $q_{2,n}^0$, above. Solving for the unknowns, we derive:

$$\text{(A.22)} \qquad q_{1,n}^0 = \frac{\log\left(\frac{A^H}{A^I}\right)(p_2^L - p_2^I) + \log\left(\frac{A^L}{A^I}\right)(p_2^I - p_2^H)}{\theta_n R_f\big[p_2^H(p_1^L - p_1^I) + p_2^L(p_1^I - p_1^H) + p_2^I(p_1^H - p_1^L)\big]},$$

$$\text{(A.23)} \qquad q_{2,n}^0 = \frac{\log\left(\frac{A^H}{A^I}\right)(p_1^I - p_1^L) + \log\left(\frac{A^L}{A^I}\right)(p_1^H - p_1^L)}{\theta_n R_f\big[p_2^H(p_1^L - p_1^I) + p_2^L(p_1^I - p_1^H) + p_2^I(p_1^H - p_1^L)\big]},$$

where $\frac{A^H}{A^I}$ and $\frac{A^L}{A^I}$ are as given in (A.20) and (A.21), that is, in terms of prices (not quantities).

Note, (A.22) and (A.23) give us ex ante equilibrium quantities $q_{1,n}^0$ and $q_{2,n}^0$ in terms of ex ante equilibrium prices p_1^0 and p_2^0 and interim equilibrium prices p_1^S and p_2^S. Next, we solve for ex ante equilibrium prices p_1^0 and p_2^0 using the market clearing condition

$$q_{i,1}^0 + q_{i,2}^0 = e, \quad i = 1, 2.$$

This yields

$$\text{(A.24)} \qquad p_i^0 = \frac{D\,K^H\,\pi(H)\,p_i^H + E\,K^L\,\pi(L)\,p_i^L + K^I\,\pi(I)\,p_i^I}{R_f(D\,K^H\,\pi(H) + E\,K^L\,\pi(L) + K^I\,\pi(I))}, \quad i = 1, 2,$$

where

$$D = \exp\left(\frac{\operatorname{erf}\big(p_1^I + p_2^I - p_1^H - p_2^H\big)\theta_1\theta_2}{\theta_1 + \theta_2}\right),$$
$$E = \exp\left(\frac{\operatorname{erf}\big(p_1^I + p_2^I - p_1^L - p_2^L\big)\theta_1\theta_2}{\theta_1 + \theta_2}\right).$$

In the above equations, $\operatorname{erf}(z)$ is the error function encountered in integrating the normal distribution, defined by

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt.$$

Next, plugging (A.24) into (A.20)–(A.21) and then using the derived expressions for $\frac{A^H}{A^I}$ and $\frac{A^L}{A^I}$ in (A.22)–(A.23), we get the desired result:[30]

$$q_{i,n}^0 = \frac{\theta_2}{\theta_1+\theta_2} \times e = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}} \times e, \quad i = 1, 2. \quad \square$$

Proof of Proposition 4. If there is homogeneous ambiguity aversion so that $\eta_1 = \eta_2$, then the interim equilibrium asset holdings given by (22) reduce to

$$q_{i,n}^S = \frac{\theta_2}{\theta_1+\theta_2} \times e = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}} \times e, \quad i = 1, 2$$

for $S = H, I, L$, which are equal to the ex ante equilibrium asset holdings as derived in Lemma A.6. Hence, the equilibrium entails no trade for any signal realization under homogeneous ambiguity aversion. Under heterogeneous ambiguity aversion, it follows from (22) that the ratio of interim asset holdings, $\frac{q_{1,n}^S}{q_{2,n}^S} = \frac{a_{1,n}^{*,S}}{a_{2,n}^{*,S}}$, depends on the signal realization S and is thus generically different from the ratio of ex ante asset holdings, $\frac{q_{1,n}^0}{q_{2,n}^0}$ (which does not depend on S as seen from Lemma A.6). $\square$

Proof of Proposition 5. To prove the result, it suffices to show that

$$\frac{\partial\left(\frac{a_{1,n}^{\star,S}}{a_{2,n}^{\star,S}} - \frac{a_{1,n'}^{\star,S}}{a_{2,n'}^{\star,S}}\right)}{\partial(\bar{\sigma}_1^S)^2} > 0$$

since $\bar{\sigma}_1^S$ is the only model parameter that varies with S. Let $\sigma = \sigma_1 = \sigma_2$. Under the assumption of the proposition,

$$\frac{a_{1,n}^{\star,S}}{a_{2,n}^{\star,S}} = \frac{(\sigma^2 + (1+\eta_n)\bar{\sigma}_2^2) - (\sigma_{12} + (1+\eta_n)\bar{\sigma}_{12})}{(\sigma^2 + (1+\eta_n)(\bar{\sigma}_1^S)^2) - (\sigma_{12} + (1+\eta_n)\bar{\sigma}_{12})}.$$

Thus,

$$\frac{a_{1,n}^{\star,S}}{a_{2,n}^{\star,S}} - \frac{a_{1,n'}^{\star,S}}{a_{2,n'}^{\star,S}} = \frac{(\eta_{n'} - \eta_n)(\sigma^2 - \sigma_{12})((\bar{\sigma}_1^S)^2 - \bar{\sigma}_2^2)}{(\sigma^2 - \sigma_{12})^2 + (2+\eta_n+\eta_{n'})(\sigma^2 - \sigma_{12})\big((\bar{\sigma}_1^S)^2 - \bar{\sigma}_{12}\big) + (1+\eta_n)(1+\eta_{n'})\big((\bar{\sigma}_1^S)^2 - \bar{\sigma}_{12}\big)^2}.$$

Note that

$$\frac{a_{1,n}^{\star,S}}{a_{2,n}^{\star,S}} - \frac{a_{1,n'}^{\star,S}}{a_{2,n'}^{\star,S}} > 0$$

for all S, because $\eta_{n'} > \eta_n$, $\sigma^2 > \sigma_{12}$, and $\bar{\sigma}_1^S > \bar{\sigma}_2$ for all S. Taking the derivative of $\frac{a_{1,n}^{\star,S}}{a_{2,n}^{\star,S}} - \frac{a_{1,n'}^{\star,S}}{a_{2,n'}^{\star,S}}$ with respect to $(\bar{\sigma}_1^S)^2$ yields:

$$\frac{(\eta_{n'} - \eta_n)(\sigma^2 - \sigma_{12})}{\Big[(\sigma^2 - \sigma_{12})^2 + (2+\eta_n+\eta_{n'})(\sigma^2 - \sigma_{12})\big((\bar{\sigma}_1^S)^2 - \bar{\sigma}_{12}\big) + (1+\eta_n)(1+\eta_{n'})\big((\bar{\sigma}_1^S)^2 - \bar{\sigma}_{12}\big)^2\Big]^2} \times$$
$$\big\{(\sigma^2 - \sigma_{12})^2 + (2+\eta_n+\eta_{n'})(\sigma^2 - \sigma_{12})((\bar{\sigma}_2)^2 - \bar{\sigma}_{12}) + (1+\eta_n)(1+\eta_{n'})((\bar{\sigma}_1^S)^2 - \bar{\sigma}_{12})(2(\bar{\sigma}_2)^2 - \bar{\sigma}_{12} - (\bar{\sigma}_1^S)^2)\big\}.$$

[30] We used Mathematica to carry out the algebraic simplifications and derive the result.

Given that $\eta_{n'} > \eta_n$ and $\sigma^2 > \sigma_{12}$, the derivative is positive if the term in the curly brackets is positive. The latter is satisfied if $(\bar{\sigma}_1)^2 < \Lambda$ where

$$\Lambda = (\bar{\sigma}_2)^2 + 2\sqrt{(\bar{\sigma}_2)^4 + \frac{(\sigma^2 - \sigma_{12})^2 + (2 + \eta_n + \eta_{n'})(\sigma^2 - \sigma_{12})((\bar{\sigma}_2)^2 - \bar{\sigma}_{12}) + (1 + \eta_n)(1 + \eta_{n'})\bar{\sigma}_{12}(\bar{\sigma}_{12} - 2(\bar{\sigma}_2)^2)}{(1 + \eta_n)(1 + \eta_{n'})}}. \tag{A.25}$$

Hence, we have the desired result. □

REFERENCES

AI, H., and R. BANSAL, "Risk Preferences and the Macroeconomic Announcement Premium," *Econometrica* 86 (2018), 1383–430.

ANDERSON, E., L. HANSEN, and T. SARGENT, "A Quarter of Semigroups for Model Specification Robustness, Prices of Risk, and Model Detection," *Journal of the European Economic Association* 1 (2003), 68–123.

ANDERSON, T., *An Introduction to Multivariate Statistical Analysis*, Wiley Series in Probability and Statistics, Volume 114 (Wiley, 1984), Hoboken, New Jersey.

BACH, L., L. CALVET, and P. SODINI, "Rich Pickings?. Risk, Return, and Skill in the Portfolios of the Wealthy," *American Economic Review* 110 (2020), 2703–47.

BACK, K., *Asset Pricing and Portfolio Choice Theory*, 2nd edition (Oxford: Oxford University Press, 2017).

BANERJEE, S., and I. KREMER, "Disagreement and Learning: Dynamic Patterns of Trade," *Journal of Finance* 65 (2010), 1269–302.

BHAMRA, H., and R. UPPAL, "Does Household Finance Matter? Small Financial Errors with Large Social Costs," *American Economic Review* 109 (2019), 1116–64.

BIANCHI, M., "Financial Literacy and Portfolio Dynamics," *Journal of Finance* 73 (2018), 831–59.

———, and J.-M. TALLON, "Ambiguity Preferences and Portfolio Choices: Evidence from the Field," *Management Science* 65 (2019), 1486–501.

BIDDER, R., and I. DEW-BECKER, "Long-Run Risk is the Worst-Case Scenario," *American Economic Review* 106 (2016), 2494–527.

BLANCHARD, O., "Movements in the Equity Premium," *Brookings Papers on Economic Activity* 2 (1993), 75–118.

BROWN, S., "The Effect of Estimation Risk on Capital Market Equilibrium," *Journal of Financial and Quantitative Analysis*, 14 (1979), 215–20.

BRUNNERMEIER, M., *Asset Pricing under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding* (Oxford:Oxford University Press, 2001).

BUSS, A., R. UPPAL, and G. VILKOV, "Investor Confidence and Portfolio Dynamics," Discussion Paper, CEPR, 2021.

CAMPBELL, J. Y., "Restoring Rational Choice: The Challenge of Consumer Financial Regulation," *American Economic Review: Papers & Proceedings* 106 (2016), 1–30.

———, *Financial Decisions and Markets* (Princeton: Princeton University Press, 2018).

———, "Asset Pricing at the Millennium," *Journal of Finance* 55 (2000), 1515–67.

CANNER, N., G. MANKIW, and D. WEIL, "An Asset Allocation Puzzle," *American Economic Review* 87 (1997), 181–91.

CAO, H., and H. OU-YANG, "Differences of Opinion of Public Information and Speculative Trading in Stocks and Options," *Review of Financial Studies* 22 (2009), 299–335.

CASKEY, J., "Information in Equity Markets with Ambiguity-Averse Investors," *Review of Financial Studies* 22 (2009), 3595–627.

CHAN, L., J. KARCESKI, and J. LAKONISHOK, "On Portfolio Optimization: Forecasting Covariances and Choosing the Risk Model," *Review of Financial Studies* 2 (1999), 937–74.

CHEW, S., M. RATCHFORD, and J. SAGI, "You Need to Recognise Ambiguity to Avoid It," *Economic Journal* 128 (2018), 2161–557.

CHIARELLA, C., R. DIECI, and X.-Z. HE, "A Framework for CAPM with Heterogeneous Beliefs," in *Nonlinear Dynamics in Economics, Finance and the Social Sciences* (Springer, 2010).353-369

CHRISTENSEN, P., and Z. QIN, "Information and Heterogeneous Beliefs: Cost of Capital, Trading Volume, and Investor Welfare," *Accounting Review* 89 (2014), 209–42.

COCHRANE, J., "Where is the Market Going? Uncertain Facts and Novel Theories," *Economic Perspectives* 21 (1997).3-37

COLLARD, F., S. MUKERJI, K. SHEPPARD, and J.-M. TALLON, "Ambiguity and the Historical Equity Premium," *Quantitative Economics* 9 (2018), 945–93.

COLLIN-DUFRESNE, P., M. JOHANNES, and L. LOCHSTOER, "Parameter Learning in General Equilibrium: Asset Pricing Implications," *American Economic Review* 106 (2016), 664–98.
CONDIE, S., J. GANGULI, and P. ILLEDITSCH, "Information Inertia," *Journal of Finance* 76 (2021), 443–79.
CUBITT, R., G. VAN DE KUILEN, and S. MUKERJI, "Discriminating between Models of Ambiguity Attitude: A Qualitative Test," *Journal of the European Economic Association* 18 (2020), 707–49.
DIMMOCK, S., R. KOUWENBERG, O. MITCHELL, and K. PEIJNENBURG, "Ambiguity Aversion and Household Portfolio Choice Puzzles: Empirical Evidence," *Journal of Financial Economics* 119 (2016), 559–77.
DRECHSLER, I., "Uncertainty, Time-Varying Fear, and Asset Prices," *Journal of Finance* 68 (2013), 1843–89.
ELLSBERG, D., "Risk, Ambiguity, and the Savage Axioms," *Quarterly Journal of Economics* 75 (1961), 643–69.
EPSTEIN, L., and M. SCHNEIDER, "Ambiguity, Information Quality, and Asset Pricing," *Journal of Finance* 63 (2008), 197–228.
FAGERENG, A., L. GUISO, D. MALACRINO, and L. PISTAFERRI, "Heterogeneity and Persistence in Returns to Wealth," *Econometrica* 88 (2020), 115–70.
GALLANT, A., M. JAHAN-PARVAR, and H. LIU, "Does Smooth Ambiguity Matter for Asset Pricing?" *Review of Financial Studies* 32 (2019), 3617–66.
GARLAPPI, L., R. UPPAL, and T. WANG, "Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach," *Review of Financial Studies* 20 (2007), 41–81.
GERBER, A., and T. HENS, "Modelling Alpha in a CAPM with Heterogenous Beliefs," *Journal of Finance and Economics* 5 (2017), 1–21.
GIANNETTI, M., and L. LAEVEN, "Local Ownership, Crises, and Asset Prices: Evidence from US Mutual Funds," *Review of Finance*, 20 (2016), 947–78.
GILBOA, I., L. SAMUELSON, and D. SCHMEIDLER, "No-Betting Pareto Dominance," *Econometrica* 82 (2014), 1405–42.
———, and D. SCHMEIDLER, "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics* 18 (1989), 141–53.
GOLLIER, C., "Portfolio Choices and Asset Prices: The Comparative Statics of Ambiguity Aversion," *Review of Economic Studies* 78 (2011), 1329–44.
HANSEN, L., and T. SARGENT, *Robustness* (Princeton University Press, 2008).Princeton, New-Jersey
———, and ———, "Fragile Beliefs and the Price of Uncertainty," *Quantitative Economics* 1 (2010), 129–62.
HARA, C., and T. HONDA, "Implied Ambiguity: Mean-Variance Inefficiency and Pricing Errors," *Management Science*, 68 (2022), 4246–60.
———, S. MUKERJI, F. RIEDEL, and J.-M. TALLON, "Efficient Allocations under Ambiguous Model Uncertainty," Working Paper 4272548, SSRN, 2022. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4272548.
HARRIS, M., and A. RAVIV, "Differences of Opinion Make a Horse Race," *Review of Financial Studies* 6 (1993), 473–506.
HE, H., and J. WANG, "Differential Information and Dynamic Behavior of Stock Trading Volume," *Review of Financial Studies* 8 (1995), 919–72.
ILLEDITSCH, P. K., "Ambiguous Information, Portfolio Inertia, and Excess Volatility," *Journal of Finance* 66 (2011), 2213–47.
JEWITT, I., and S. MUKERJI, "Ordering Ambiguous Acts," *Journal of Economic Theory* 117 (2017), 213–67.
JU, N., and J. MIAO, "Ambiguity, Learning, and Asset Returns," *Econometrica* 80 (2012), 559–91.
KANDEL, E., and N. PEARSON, "Differential Interpretation of Public Signals and Trade in Speculative Markets," *Journal of Political Economy* 103 (1995), 831–72.
KIM, O., and R. VERRACHIA, "Market Liquidity and Volume around Earnings Announcements," *Journal of Accounting and Economics* 17 (1994), 41–67.
KLEIN, R., and V. BAWA, "The Effect of Estimation Risk on Optimal Portfolio Choice," *Journal of Financial Economics* 3 (1976), 215–31.
KLIBANOFF, P., M. MARINACCI, and S. MUKERJI, "A Smooth Model of Decision Making under Uncertainy," *Econometrica* 73 (2005), 1849–92.
———, ———, and ———, "Recursive Smooth Ambiguity Preferences," *Journal of Economic Theory* 144 (2009), 930–76.
KNIGHT, F., *Risk, Uncertainty and Profit* (Houghton Miffin, 1921).Boston and New York
KOSTOPOULOS, D., S. MEYER, and C. UHR, "Ambiguity about Volatility and Investor Behavior," *Journal of Financial Economics*, 145 (2022), 277-96.
LETTAU, M., and S. C. LUDVIGSON, "Measuring and Modeling Variation in the Risk-Return Trade-Off," in *Handbook of Financial Econometrics* (Amsterdam Elsevier B.V., 2010), 617–90.
LUDVIGSON, S. C., "Advances in Consumption-Based Asset Pricing: Empirical Tests," in *Handbook of the Economics of Finance* (Amsterdam Elsevier B.V., 2012), 799–906.

MACCHERONI, F., M. MARINACCI, and D. RUFFINO, "Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis," *Econometrica* 81 (2013), 1075–113.

MAENHOUT, P., "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies* 17 (2004), 951–83.

MARKOWITZ, H., "Portfolio Selection," *Journal of Finance* 7 (1952), 77–91.

MELE, A., and F. SANGIORGI, "Uncertainty, Information Acquisition, and Price Swings in Asset Markets," *Review of Economic Studies* 82 (2015), 1533–67.

MERTON, R., "On Estimating the Expected Return on the Market," *Journal of Financial Economics* 8 (1980), 323–61.

MONGIN, P., "Consistent Bayesian Aggregation," *Journal of Economic Theory* 66 (1995), 313–51.

———, "Spurious Unanimity and the Pareto Principle," *Economics and Philosophy* 32 (2016), 511–32.

PIKETTY, T., *Capital in the Twenty-First Century* (Harvard University Press, 2014). Cambridge, MA

RIGOTTI, L., C. SHANNON, and T. STRZALECKI, "Subjective Beliefs and Ex-Ante Trade," *Econometrica* 76 (2008), 1167–90.

RUBINSTEIN, M., "An Aggregation Theorem for Securities Markets," *Journal of Financial Economics* 1 (1974), 225–44.

RUFFINO, D., "A Robust Capital Asset Pricing Model," Finance and Economics Discussion Series 2014-01, Board of Governors of the Federal Reserve System (U.S.), 2014.

SCHMEIDLER, D., "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57 (1989), 571–87.

STRZALECKI, T., and J. WERNER, "Efficient Allocations under Ambiguity," *Journal of Economic Theory* 146 (2011), 1173–94.

TOBIN, J., "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies* 25 (1958), 65–86.

TRAUTMANN, S. T., and G. VAN DE KUILEN, "Ambiguity Attitudes," in G. Keren and G. E. Wu, eds., *The Wiley Blackwell Handbook of Judgment and Decision Making*, (Blackwell, 2015), 89–116.

WERNER, J., "Speculative Trade under Ambiguity," *Journal of Economic Theory* 199 (2022), 105200.